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NEW METHODS IN GEOMETRICAL OPTICS

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NEW METHODS IN GEOMETRICAL OPTICS

WITH SPECIAL REFERENCE TO THE DESIGN OF CENTERED OPTICAL SYSTEMS

BY

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PREFACE

The ordinary practice in designing an optical system is to trace a sufficient number of rays of light through the various lenses of assumed radii, thicknesses and separations by trigonometric calculation, then, modifying these elements in accordance with experience, the process is repeated until the designer is satisfied that the errors still indicated are so small that the working optician will be able to eliminate them by empirical methods.

In this process the fundamentals of the calculations are "light-rays," "radii" and "indices of refraction," no one of which is a physical entity. Moreover, the calculations do not lead to a direct determination of the quantities of first importance. For example, no astronomer cares whether the focal length of his telescope departs from the prescribed value by one part in a thousand or even ten times that amount, but errors in color correction or in zonal differences of focal distances one tenth as great would be fatal to the highest excellence. Obviously, what the designer really lacks is a method which would give at once these secondary errors, while he would readily sacrifice something of the useless precision in the primary quantities for a greater facility. Stated in other words, he would find derivative equations far more useful if not too complicated; but such equations as derived from the familiar equa-

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tions of optics are not simple enough to be generally useful. It follows from this that a designer of complicated optical systems is constrained to go through a great amount of labor of which much is of no ultimate value.

If we could replace the above named fundamentals, which are not physical entities because they cannot be defined either in place or in time, by quantities which are free from this disqualification, we might expect a considerable gain in the rational discussion of optical problems even if in no other respect. But in fact the advantages attained by such a substitution are quite astonishingly great. Not only are the fundamental equations so simplified that their derivatives are easily found but the labor required in designing a lens system is reduced by an incredible amount.

The following text may be regarded as a development of the consequences of introducing the real entities — wave-surfaces, surface curvatures and wave-velocities — in place of the data ordinarily employed.

Finally, it may properly be stated that some of the propositions derived here I have already published in 1893 in Trans. Amer. Acad. Sci., Vol. VI, and, to a greater extent, in the book cited on p. 94 below. In neither of these publications, however, was emphasis placed upon their practical utility which has been abundantly tested by me during the many years that Dr. J. A. Brashear and his successor, J. B. McDowell, engaged my aid in their numerous problems.

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NEW METHODS IN GEOMETRICAL OPTICS

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CHAPTER I

General Equations

In a general discussion of the properties of an optical system we are obliged to adopt certain conventions which define the magnitudes employed. In the considerations which follow we shall assume that the refracting, or reflecting, surfaces are spherical and that their geometrical centers lie on a straight line which is called the axis of the system. Moreover, we shall suppose, except in cases specifically designated, that the light is propagated from left to right in our diagrams and that spherical surfaces are defined by their curvatures, the surfaces being regarded as positive when their geometrical centers lie to the right of the vertices where the axis of the system intersects them.

With these postulates it is easy to find the effect of any number of refracting surfaces upon wave-surfaces which have their centers on, or very near to, the axis, as appears in the following section.

In Fig. 1, let p, o' and o_1 be, respectively, the positions of the geometrical centers of a refracting surface of curvature γ, of an incident wave-surface of curvature c', and of a refracted wave-surface of curvature c_1.

Let k be the indefinitely short length of a common semi-chord to the three arcs in the figure; then, if x, x', and x_1 are the sagittas of these arcs we have the values,

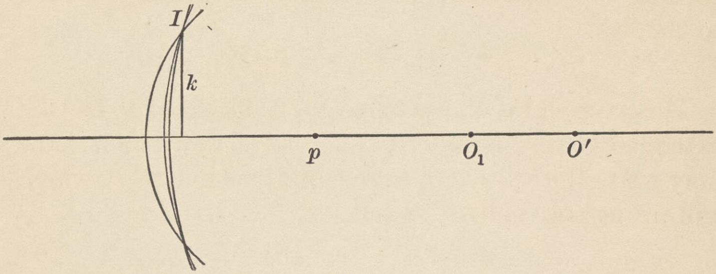

Fig. 1

$$x = \tfrac{1}{2}k^2\gamma;\ x' = \tfrac{1}{2}k^2c';\ x_1 = \tfrac{1}{2}k^2c_1.$$

From the laws of wave propagation we have, if ρ is the ratio of the velocity of light in the medium to the right of the surface γ to that in the medium to the left,

$$x - x_1 = \rho(x - x').$$

Substituting the above values of x, x' and x_1 and solving for c_1 we have,

$$c_1 = \gamma(1 - \rho) + \rho c'.$$

This wave-surface will be propagated with uniform velocity and uniformly decreasing radius until it reaches a second refracting surface at a distance t_1 from the first, when its radius is reduced from $\dfrac{1}{c_1}$ to $\dfrac{1}{c_1} - t_1$, and its curvature is increased to the reciprocal of this last quantity, namely, to

$$c_1(1 - c_1t_1)^{-1} = \mu_1c_1,$$

if we agree to define μ by this equation. After refraction at the second surface, the curvature will be, according to exactly the same reasoning as that applied to the first,

$$c_2 = \gamma'(1 - \rho_1) + \mu_1\rho_1c_1.$$

This completes the general solution, since it can be extended to any number of successive refractions. We may write the system of such equations in a form which will prove convenient, as follows:

$$\begin{array}{llll}
c_1 & = \gamma(1 - \rho) + \rho c' & c' & = c' \\
c_2 & = \gamma'(1 - \rho_1) + \rho_1 c'' & c'' & = \mu_1 c_1 = (1 - c_1t_1)^{-1}c_1 \text{ (a)} \\
c_3 & = \gamma''(1 - \rho_2) + \rho_2 c''' & c''' & = \mu_2 c_2 = (1 - c_2t_2)^{-1}c_2 \\
 & \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
c_{\lambda+1} & = \gamma^\lambda(1 - \rho_\lambda) + \rho_\lambda c^{\lambda+1} & c^{\lambda+1} & = \mu_\lambda c_\lambda = (1 - c_\lambda t_\lambda)^{-1}c_\lambda
\end{array}$$

If the lenses of the system are indefinitely thin and in contact — in other words, if all the t's are equal to zero and hence all the μ's equal to unity — the group (a) can be replaced by a single equation, namely,

$$c_{\lambda+1} = \gamma^\lambda(1 - \rho_\lambda) + \gamma^{\lambda-1}(1 - \rho_{\lambda-1})\, \rho_\lambda \ldots + \gamma(1 - \rho)\\ (\rho_\lambda\rho_{\lambda-1} \ldots \rho_1) + \rho_\lambda\,!\,c'$$

Now let n^0 be the index of refraction of the first medium, n' that of the second, and so on, then the values of the velocity ratios are as follows:

$$\rho = \frac{n^0}{n'},\ \rho_1 = \frac{n'}{n''},\ \ldots\ \rho_\lambda = \frac{n^\lambda}{n^{\lambda+1}},$$

and the continued product, $\rho_\lambda!$ will equal $\frac{n^0}{n^{\lambda+1}}$ which is the ratio of the velocity of light in the last medium to that in the first. This quantity, important in the theory of images, we shall designate by ρ_0. By means of this notation we may write the equation above in the condensed form:

$$c_{\lambda+1} = P + \rho_0 c'.$$

II. Images and Magnification

When the geometrical center of a system of wave-surfaces is a source of light, this source is styled, in the language of optics, the object, and the geometrical center of the system as modified by refraction, or otherwise, is called its image. We shall extend slightly this use of the words so that the geometrical center of a system of wave-surfaces, wherever placed, before modification is called the object, after modification the image. Thus, every center of the system of wave-surfaces defined by equations (a) is either object or image, according to whether it is regarded as pertaining to the following refracting surface or to the preceding one.

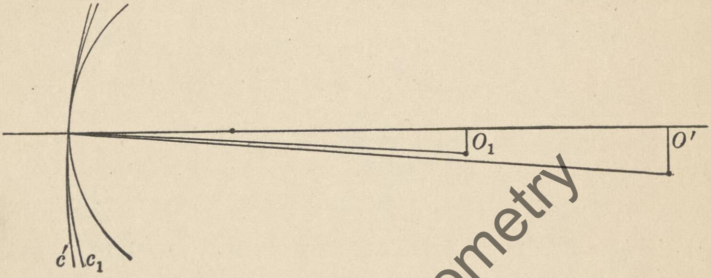

Fig. 2

Let the center of the incident wave-surface be displaced by the indefinitely small distance o' (Fig. 2), then the center of c_1 will also be displaced a certain small distance which we shall designate as o_1. The line o_1 is, from the principle of continuity, the optical image of the line o'; moreover, from the law of refraction, the angle subtended by o_1 at the vertex of γ is ρ times that subtended by o', whence

$$\frac{o_1}{o'} = \rho \frac{c'}{c_1}$$

This quantity may be called the magnification due to refraction at the first surface, and to extend it to a more complicated system we have only to extend the reasoning to all the surfaces, thus, for $\lambda + 1$ surfaces,

$$\frac{o_1}{o'} = \rho \frac{c'}{c_1}$$

$$\frac{o_2}{o_1} = \rho_1 \frac{\mu_1 c_1}{c_2}$$

$$\cdots\cdots\cdots$$

$$\frac{o_{\lambda+1}}{o_\lambda} = \rho_\lambda \frac{\mu_\lambda c_\lambda}{c_{\lambda+1}}$$

The ratio of the final image to the initial image, or the magnification of the system, is the continued product of the above values, namely,

$$\frac{o_{\lambda+1}}{o'} = \rho_0 \mu_\lambda! \frac{c'}{c_{\lambda+1}}$$

The value of $\mu_\lambda!$ can be readily determined by calculation in a given system, but a general analytical expression for it derived from equations (a) could hardly be looked for since the equations do not explicitly contain the essential limitation that they only hold true in the immediate vicinity of the axis. It is obvious that this value is a function of the variable c' and of all the physical constants of the system, but the feature of importance here is the form in which this variable

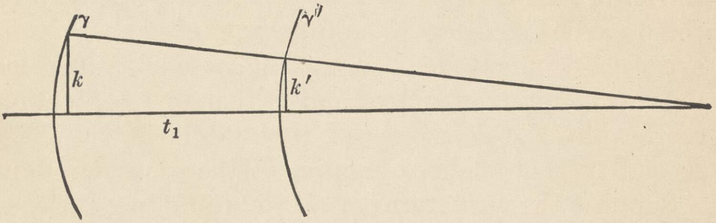

Fig. 3

enters. To ascertain this let γ and γ' (Fig. 3) be the first and second refracting surface, respectively, then

$$k' : k = \frac{1}{c_1} : \frac{1}{c_1} - t_1, \text{ whence } k' = k(1 - c_1 t_1)$$

Extending this method to the successive surfaces we may write:

$$\begin{aligned} k' &= k(1 - c_1 t_1) \\ k'' &= k'(1 - c_2 t_2) \quad \therefore k^\lambda = k(1 - c_1 t_1)(1 - c_2 t_2) \ldots (1 - c_\lambda t_\lambda) \\ &\ldots\ldots \\ k^\lambda &= k^{\lambda-1}(1 - c_\lambda t_\lambda) \qquad = \mu_\lambda^{-1}!\, k \end{aligned}$$

The modification which a wave-surface undergoes in passing from one medium to another consists of two parts, both involving the change of velocity of propagation, but the first in addition to this only the curvature of the interface and the second that of the wave-surface itself. When all the products kc_1, kc_2, etc., are very small we may regard the change in length between two successive values of k as equal to the sum of the changes due to each of the effects named above; moreover, we may, always with the limitation as regards departures from the region of the axis, consider the total change from k to k^λ as equal to the sum of these partial changes. The first type of changes is in every case independent of the value of curvature of the incident wave-surfaces and is therefore a function of the physical constants of the system alone. The second type of changes requires further consideration.

Suppose the curvatures of all the refracting surfaces to be zero and consider the successive increments of k due to the value of c'. The inclination of that portion of the incident wave-surface at the distance k from the axis to the refracting surface would be kc', changed by refraction to $k\rho c'$; the increase of k' over k would be $-k\rho c' t_1$. Just so, after refraction at the second surface, the inclination of the portion of the wave-surface in question would be $k\rho\rho_1 c'$, whence the increase of k'' over k' would be $-k\rho\rho_1 c' t_2$. Extending the reasoning to the whole system of surfaces we find that the whole change from k to k^λ due to c', that is to say, the sum of the above terms, is a linear function of kc', hence the increment of length due to the first effect may be set equal to bk and that due to the second effect as equal to $\beta c' k$, where b and β are constants depending on the physical constants of the system alone. The sum of the two effects taken together is embodied in the equation

$$k^\lambda - k = k(b + \beta c'),\text{ or, finally, } k^\lambda = k(\alpha + \beta c'),$$

where α and β depend solely on the physical constants of the system. Comparing this expression for k^λ with the one above and noting the values of μ_1, etc., as given in equations (a) we see that

$$\mu_\lambda! = (\alpha + \beta c')^{-1}$$

and that the expression for the magnification of the system becomes

$$\frac{o_{\lambda+1}}{o'} = \rho_0 \frac{c'}{c_{\lambda+1}(\alpha + \beta c')}. \tag{b}$$

There is another very important expression for the magnification of the system which may be derived as follows:

Suppose that there is a circular diaphragm at the first surface with the small radius k. Let the semi-angular diameters of the wave-surfaces, incident and refracted, thus limited, be designated by ω' and ω_1, and so on for successive refracting surfaces; then

$$\begin{aligned}
\sin \omega' &= kc' \\
\sin \omega_1 &= kc_1 \\
\sin \omega_2 &= k'c_2 \\
\cdot\ \cdot\ \cdot\ \cdot &\quad \cdot\ \cdot\ \cdot\ \cdot \\
\sin \omega_\lambda &= k^{\lambda-1}c_\lambda \\
\sin \omega_{\lambda+1} &= k^\lambda c_{\lambda+1}
\end{aligned}$$

From these equations we can write, in regarding the relation of the values of successive k's as developed on page 6,

$$\begin{array}{ll}
\dfrac{\sin \omega'}{\sin \omega_1} = \dfrac{c'}{c_1} & \quad \cdot\ \cdot\ \cdot\ \cdot \\[2ex]
\dfrac{\sin \omega_1}{\sin \omega_2} = \mu_1 \dfrac{c_1}{c_2} & \quad \dfrac{\sin \omega_\lambda}{\sin \omega_{\lambda+1}} = \mu_\lambda \dfrac{c_\lambda}{c_{\lambda+1}}.
\end{array}$$

The continued product of these ratios is

$$\frac{\sin \omega'}{\sin \omega_{\lambda+1}} = \mu_\lambda! \frac{c'}{c_{\lambda+1}}$$

Multiplying both sides of the last equations by ρ_0 and observing (b), we have, as another expression for the magnification of the system,

$$\frac{o_{\lambda+1}}{o'} = \rho_0 \frac{\sin \omega'}{\sin \omega_{\lambda+1}} \qquad (c)^*$$

III. SIMPLIFICATION OF EQUATIONS

In the foregoing discussion the curvature of the incident wave-surface has been that possessed when attaining the first vertex of the system, while that of the finally refracted wave-surface is proper to the last vertex. These two points are perfectly convenient for expressing the results of direct calculations, or of measurements such as the lens-maker employs, but it is not to be expected that these particular points of reference should possess signal advantages when the system is regarded in any other light. It is important, therefore, to find what forms the equations may take when other points of reference are chosen, in short, to find a system of transformation equations. To do this is the aim of this section.

Let the new points of reference, Fig. 4, be at a distance ξ and ξ', respectively, from the first and last vertices of the system. In addition, let the incident wave-surfaces be bounded by a circular diaphragm somewhere so that at ξ the wave-surface will have the semi-diameter a and, as a consequence, a definite value after final refraction which, at ξ', may be denoted by a'; then, if C' and C_1 represent the curvatures of the two wave-surfaces under consideration at ξ and ξ' the following relations are obvious:

---

* See Appendix A.

$$\begin{aligned} c' &= C'(1+\xi C')^{-1} & \qquad c_{\lambda+1} &= C_1(1+\xi' C_1)^{-1} \\ C' &= c'(1-\xi c')^{-1} & \qquad C_1 &= c_{\lambda+1}(1-\xi' c_{\lambda+1})^{-1} \end{aligned} \qquad \text{(d)}$$

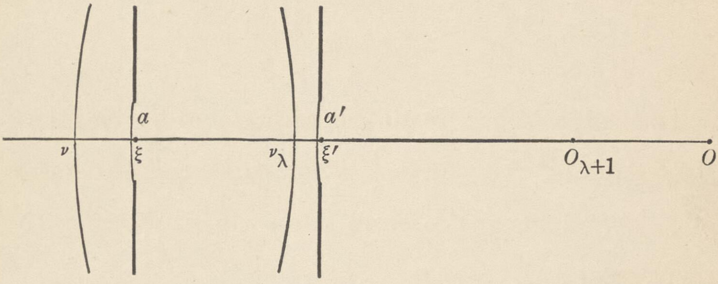

Fig. 4

These are the desired transformation equations. In addition

$$\sin\omega' = aC', \ \sin\omega_{\lambda+1} = a'C_1.$$

The last equations, with (b) and (c), yield

$$\frac{a}{a'}\frac{C'}{C_1} = \frac{c'}{c_{\lambda+1}(\alpha+\beta c')}.$$

A marked simplification in the expression for the relation of C' and C_1 is gained if ξ' is so chosen that it is the optical image of ξ, for then a' is the optical image of a and the ratio $\frac{a}{a'}$ becomes independent of c'. This ratio will be designated by g. With this condition adopted, the last equation, by means of the transformation equations (d), may be written,

$$C_1 = \frac{g\alpha - 1}{\xi'} + \frac{g(\alpha\xi+\beta)}{\xi'}C' \qquad \text{(e)}$$

The first term on the right in (e) is the value of C_1 when C' (and therefore also c') equals zero. Designating this particular value of C_1 by C_1^0, and, similarly, the value of $C_{\lambda+1}$ when c' equals zero by $c_{\lambda+1}^0$, the last equation but one combined with (e) yields

$$C_1^0 = g\alpha c_{\lambda+1}^0 = \frac{g\alpha - 1}{\xi'}.$$

The value of g is readily determinable by means of (b) by taking c' equal to $\frac{1}{\xi}$, when, since ξ' is the image of ξ, $c_{\lambda+1}$ equals $\frac{1}{\xi'}$. Making these substitutions in (b) there results

$$\frac{1}{g} = \frac{\rho_0\xi'}{\alpha\xi + \beta}.$$

The equation (e) may now be written

$$C_1 = g\alpha c_{\lambda+1}^0 + \rho_0 g^2 C', \tag{e$'$}$$

a remarkably simple equation to replace the system of equations (a). In this equation only C_1 and C' are variable; g is a number, arbitrarily chosen, and the other terms depend only on the physical constants of the system.

The places of the new points of reference, referred to the last and to the first vertices of the system, respectively, are immediately derived from the foregoing relations, namely:

$$\xi' = \frac{g\alpha - 1}{g\alpha c_{\lambda+1}^0}; \ \xi = \frac{g\rho_0\xi' - \beta}{\alpha} \tag{f}$$

IV. Extension of Conception of Magnification

In Section II the magnification of an optical system was defined as the ratio of the displacement of the image of a point from the axis to that of the point source, or object. This, the ordinary conception of magnification, may be specified as the *transverse* magnification; its value, which we shall designate as M, can readily be expressed in terms of C' and C_1 by means of (b) and the equation

$$\frac{a}{a'}\frac{C'}{C_1} = \frac{c'}{c_{\lambda+1}(\alpha + \beta c')}.$$

They give

$$M = \rho_0 g \frac{C'}{C_1} \qquad \text{(g)}$$

The definition of the transverse magnification suggests a second kind of magnification, namely the ratio of the displacement of the image along the axis to a corresponding displacement of the object. This type of magnification we may style the *longitudinal* magnification and designate by L. From the definition and (e′) we find

$$L = \frac{d(1/C_1)}{d(1/C')} = \frac{dC_1}{dC'}\frac{C'^2}{C_1^2} = \rho_0 g^2 \frac{C'^2}{C_1^2}.$$

Combining this result with (g) the important relation connecting the two kinds of magnification is established in the equation

$$L = \frac{M^2}{\rho_0} \qquad \text{(h)}$$

In many optical systems the value of M is not discriminative; for example, in all telescopes adjusted for vision at infinity, that is to say, where both object and image are at an infinite distance, M is indeterminate; for all cameras adjusted for objects at an infinite distance M is equal to zero, and, finally, for all systems where the object is at the first principal focus M is equal to infinity. In such cases as well as in cases approximating to them, the expression for the *angular* magnification is useful. This, which we shall designate by N, is defined as the ratio of the angular subtense of the image as measured from ξ' to that of the object as measured from ξ, therefore

$$N = \frac{o_{\lambda+1}C_1}{o'C'} = M\frac{C_1}{C'},$$

or, in view of (g),

$$N = \rho_0 g \qquad \text{(i)}$$

The equation (h), on account of its perfect generality, merits attention which is as appropriately given here as elsewhere. It follows immediately from it that the total depth of the region through which the images near the axis are distributed is very small when the nearest object is relatively remote and M is small. Thus, in the case of the photographic camera used in ordinary out of doors practice, ρ_0 is unity and L is so small that in short cameras provision for altering the length may be omitted without materially reducing their utility. The quite irrational term "universal focus" has its origin in this practice.

Again, in the case of the microscope the value of M is large, hence the longitudinal magnification is much greater. This explains why the total range of depth of an object which can be seen without variation of adjustment is so small; it also shows why an object looks so much flatter when imbedded in a transparent medium of high refractive index, ρ_0 then being equal to this number.

V. On Particular Values of g in the General Equations

Inspection of equation (e′), that is,

$$C_1 = g\alpha c^0_{\lambda+1} + \rho_0 g^2 C',$$

shows that when C' is equal to zero $\frac{1}{C_1}$ is the distance from ξ' to the axial image of a point at an infinite distance from ξ; the point thus defined is called the second principal focus of the system and will be designated hereafter by the symbol f'. Conversely, when C_1 is equal to zero $\frac{1}{C'}$ is the distance from ξ to the point whose image is at an infinite distance. This point is called the first principal focus and will be designated by f.

All the foregoing values, except the places of the points f and f', are dependent on the arbitrary choice of the number g, which choice, in the abstract, is not important; there are, however, certain obvious values of g which have practical advantages. Such are 1, -1, $\frac{1}{\rho_0}$ and $-\frac{1}{\rho_0}$. Taking these in turn we have four sets of values for the variables defined by the fundamental equations (e′), (g), (h) and (i). These may be grouped as follows:

I
$(g = 1)$
Principal points

$$\begin{cases} C_1 = \alpha c^0_{\lambda+1} + \rho_0 C' \\ M = \rho_0 \dfrac{C'}{C_1} \\ L = \rho_0 \left[\dfrac{C'}{C_1}\right]^2 \\ N = \rho_0 \end{cases}$$

II
$(g = -1)$
Negative principal points

$$\begin{cases} C_1 = -\alpha c^0_{\lambda+1} + \rho_0 C' \\ M = -\rho_0 \dfrac{C'}{C_1} \\ L = \rho_0 \left[\dfrac{C'}{C_1}\right]^2 \\ N = -\rho_0 \end{cases}$$

III
$\left(g = \dfrac{1}{\rho_0}\right)$
Nodal points

$$\begin{cases} \rho_0 C_1 = \alpha c^0_{\lambda+1} + C' \\ M = \dfrac{C'}{C_1} \\ L = \dfrac{1}{\rho_0}\left[\dfrac{C'}{C_1}\right]^2 \\ N = 1 \end{cases}$$

IV
$\left(g = -\dfrac{1}{\rho_0}\right)$
Negative nodal points

$$\begin{cases} \rho_0 C_1 = -\alpha c^0_{\lambda+1} + C' \\ M = -\dfrac{C'}{C_1} \\ L = \dfrac{1}{\rho_0}\left[\dfrac{C'}{C_1}\right]^2 \\ N = -1 \end{cases}$$

No one of these groups is superior to the others in point of simplicity or of utility. In a large class of instruments, by far the largest brought to the attention of the optician, the first and last media are optically equivalent, whence ρ_0 equals unity, and in all such cases Groups I and III become identical as do also Groups II and IV.

Insight as to the physical meaning of the above sets of equations may be obtained by discussing Groups I and II when ρ_0 is unity, the familiar case, and then Groups III and IV when ρ_0 has a different value, as, for example, it has in the case of the eye.

From I, $C_1 = \alpha c^0_{\lambda+1} + C'$, which is exactly the form of the expression for infinitely thin lenses in contact, $\alpha c^0_{\lambda+1}$, being the power of the system. A small object at ξ has its image at ξ' of the same size and orientation, since $M = g = 1$ and $L = 1$ also. These are the only points on the axis where these conditions obtain; they were named the first and second principal points by Gauss, who discovered their properties. We shall employ the symbols e and e' to indicate them. When C' is equal to zero, C_1 is equal to the power of the system, and its reciprocal is the distance, from the second principal point to f', the second principal focus. When C_1 equals zero, C' equals the power taken negatively, and its reciprocal equals the distance from the first principal point to the first principal focus. The angular magnification is equal to unity for all positions of the object.

From Group II, $C_1 = - \alpha c^0_{\lambda+1} + C'$. When C' equals zero, $C_1 = - \alpha c^0_{\lambda+1}$, and the reciprocal of this quantity is the distance from ξ' to f'; so too, when C_1 equals zero the reciprocal of C' is the distance from ξ to the first principal focus. A small object at ξ has an inverted image at ξ' of the same linear dimensions as the object, since $M = g = - 1$ and $L = M^2 = 1$. These are the only points on the axis where such relations obtain; they may be called, by analogy, the first and second negative principal points. When we wish to designate these particular values of ξ and ξ' we shall use the symbols $\underline{e}$ and $\underline{e}'$. The angular magnification is, for every position of the object, equal to -1, which means that, if the object as seen from $\underline{e}$ is regarded as erect, the image as seen from $\underline{e}'$, facing the same direction, is inverted but otherwise of the same angular dimensions. It will be observed that f' falls exactly halfway between e' and $\underline{e}'$ just as f lies midway between e and $\underline{e}$; thus it follows that two of the three pairs of points being given, the remaining two can be deduced at once without other calculation.

Points on the axis such as defined above are called, generically, *cardinal points*. If a sufficient number of the cardinal points are given, the geometric construction of the image corresponding to any object is very simple. To illustrate this fact, let Fig. 5 represent the axis of an optical system with its cardinal points as indicated. Suppose it is required to construct the image of the object abc; proceed as follows: Consider light, originating at a, moving towards e; after final refraction it will move in an unchanged direction but the energy which, before incidence, would pass through e now passes through e', since the latter point is the

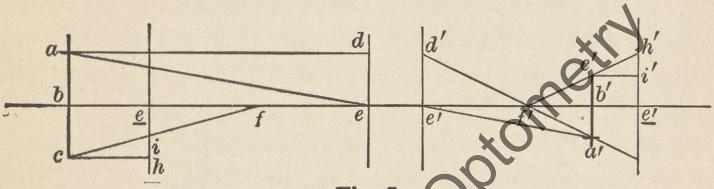

Fig. 5

image of the former. Let $e'a'$ be this line of propagation from e', then the image of a must lie on the line. Again, consider light waves, originating at a, which are propagated before incidence in a direction parallel to the axis; after final refraction the line of propagation passes through f' and also through d' which is the image of d. The image of a lies, therefore, on the point of intersection of the two lines drawn in the image region at a'.

An equally simple method of constructing the image is to use the negative principal points with the principal focal points. This is shown in the figure as applied to the finding of the image of c, where the line ch is drawn in the object region parallel to the axis, and then the corresponding line $f'h'$ in the image region, h' being the image of h. Next, draw cf in the object region and the corresponding line $c'i'$ the image region, i' being the image of i, then the point common to the two lines in the image region is the position of the image of c. It is obvious that in either of the cases above considered the points f and f' could be replaced by the pair of principal points without rendering the construction less simple.

It is easier, and incomparably more accurate, to calculate the required values by means of the equations I or II. Thus, in the present case, if we fix the scale by setting $e'f'$ as equal to unity, we have $\frac{1}{C'} = -2.75$ (by arbitrary assumption) whence, from I, $\frac{1}{C_1} = 1.571$, the distance from e' to b'. From I also, we have M equal to $-\frac{1.571}{2.75}$ which is equal to the ratio of the length $a'b'$ to ab. Equally direct is the calculation by equations II, where $\frac{1}{C_1}$ equals -0.429, since $\frac{1}{C'}$ equals -0.75. The second equation of II gives the value of M equal to -0.571, the same as that calculated from I. If the problem had not been specially chosen to illustrate the graphical method the advantages of the analytical method would be even more striking. Indeed, it may be asserted that the value of a knowledge of the principal points of a system has been much overestimated by writers on optical instruments; they have failed to note that these points are of little practical value except as an aid in diagrammatic description.

As an example of the case where the first and last media are unlike, we will suppose that the first is air and the last is water with an index of refraction equal to $\frac{4}{3}$, whence the value of ρ becomes equal to $\frac{3}{4}$. From Group III we find the relation of C_1 to C' expressed by the equation $\frac{3}{4} C_1 = \alpha c^0_{\kappa+1} + C'$, whence the first principal focus lies to the left of ξ at a distance equal to the reciprocal of the power of the system; the second principal focus lies to the right of ξ' at a distance $\frac{3}{4}$ as great. A small object at ξ has its erect image at ξ', but its transverse dimensions are decreased in the ratio $\frac{3}{4}$ while its longitudinal dimensions are decreased in the ratio of the square of this number. The fourth equation of the group shows that the angular value of the image as measured from ξ' is equal to the angular value of the object as measured from ξ. This last property of the two reference points seemed of particular significance to its discoverer, Listing, and he named them *nodal points*. When we refer to them we shall, following Listing, name them the first and second nodal points and designate them by n and n', respectively.

The first equation of Group IV shows that the first principal focus lies to the right of the first reference point at a distance equal to the reciprocal of the power; the second principal focus lies to the left of the second reference point at a distance $\frac{3}{4}$ as great. A small object at ξ has an image at ξ' with transverse magnification of $-\frac{3}{4}$ and a longitudinal magnification represented by the square of this number. The image is therefore inverted. The last equation shows that the angular subtense of the image as measured from ξ' is numerically equal to that of the object as measured from ξ. This property attaching to this pair of reference points suggests the name of *negative nodal points*, which we may designate by $\underline{n}$ and $\underline{n}'$, respectively.

The foregoing results deduced from the groups of equations III and IV give an easy method for a geometrical construction to the image of a given object, when the cardinal points proper to the system are known. This is shown in Fig. 6. Here the construction for the point a' employs the nodal points and the second principal focus, the distance nd being g times the distance

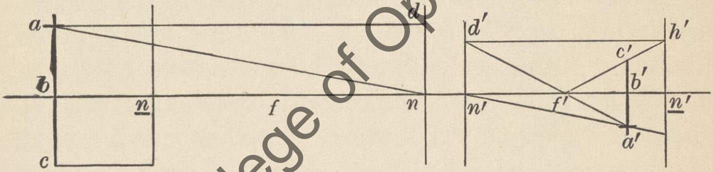

Fig. 6

$n'd'$; an alternative construction is chosen for c' where the negative nodal points and f' are employed, the distance $\underline{n}'h'$ being $-g$ times $n'h$.*

Should one desire to find the principal points also, it is necessary to have recourse to Group I with a value of ρ_0 equal to $\frac{3}{4}$.

A glance at these equations when ρ_0 is given its proper value shows that the principal points, separated by the same interval as that separating the nodal points, are symmetrically placed as regards the latter with respect to the principal focal points.

The above discussion of the interpretation of the groups shows that, in general, having given the positions of any two pairs of cardinal points, the positions of the remaining three pairs are at once known.

VI. THE DETERMINATION OF CARDINAL POINTS

The problem of determining the position of the cardinal points is one which ordinarily presents itself in one of two forms: either all the elements of the system are known, in which case we can find the required points by simple calculation, or a finished system of more or less completely unknown construction is given when an experimental method of fixing the principal points is required. The first method is the one which the designer and the constructor of optical apparatus has at command, the second is that to which the user of such apparatus, especially if complicated, is generally restricted. A short discussion of the two methods will prove advantageous.

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* The construction was, as a matter of fact, made in the following far more accurate and easier manner. Assumptions: $C' = -\frac{1}{2.75}$, $P = 1$; then $\frac{1}{C_1} = 1.179$ and $M = -0.429$. After object and image were drawn in indicated places and indicated ratios of length, the illustrative construction lines were drawn.

In order to find the positions of e and e' we have only to substitute the proper value of g, here unity by definition, in equations (f), page 11. Thus:

$$\xi' = \frac{\alpha - 1}{\alpha c^0_{\lambda+1}}; \ \xi = \frac{\rho_0 \xi' - \beta}{\alpha} \qquad \text{(j)}$$

where ξ' is the distance from the final vertex to the second principal point and ξ that from first vertex to the first principal point. To solve these equations calculate the value of $c^0_{\lambda+1}$ by means of equations (a); the reciprocal of this value is the distance from the last vertex of the system to the second principal focus and α^{-1} is equal to $\mu_\lambda!$, thus the positions of e' and f' are determined. To determine e and f there are two methods suggested; the calculations of (a) may be repeated with any value of c' other than zero when the value of $\mu_\lambda!$ will be $(\alpha + \beta c')^{-1}$, whence β can be derived and the position of e calculated with the dependent position of f. Another procedure would be to carry the computations of (a) in the reverse order with $c^0_{\lambda+1}$ assumed equal to zero; the reciprocal of the resulting value of c' will be the distance from the first vertex to the first principal focus, whence the position of e would be determined, and thus all the cardinal points.

The general problem in its other aspect is, of course, extremely varied. To set some convenient limit to the conditions we may assume that ρ_0 is equal to unity as this will, at any rate, except from consideration only a small number of forms. In these cases Group I affords the convenient relations for measurement. In very many cases the positions of the principal foci, referred to a point in the lens system or to a point in its mounting, can very readily be found. Less frequently, the positions of the principal points can be found by an experimental shifting of the object until its image is of the same size and erect; this method is practicable for the compound microscope. When this fails, a determination of the positions of the negative principal points is generally simple. In systems of high power, like microscope objectives, a ready method is to measure the length of the image of a distant object whose angular value is known, for then, the angular magnification of the system being unity, we may find the distance separating the focal point from the nearer principal point and, by referring the position of the image to a fixed point in the system, both points are established. A repetition of the measures with the objective reversed will give the other points similarly related. This method is one of the best to employ in determining the absolute focal length of a system of so low a power as the objective of an astronomical telescope, since it is easy to find the angular value of a fixed length in the principal focal plane by transits of stars of known declination; the difference between this value and the focal distance is the measure of the distance from the second principal point to the rear vertex.

Occasionally the fourth equation in Group I may be conveniently used to find the principal points, since it shows that when the object is at an infinite distance the image remains fixed in position if the system is rotated about an axis through the nearer principal point. If the system is mounted so as to slide upon a rotating carrier, the axis of rotation being perpendicular to the optical axis, this method may be quite accurate as well as convenient, since apparent displacements of images suffer reversal of direction when the principal point crosses the axis of rotation.

CHAPTER II

Examples Illustrating Applications of General Equations — Complete Theory of Plane Mirror — Uses and Limitations of Cardinal Points — Telescope and Microscope

In order to illustrate the utility of cardinal points, as well as the equally important limitations in their use, we shall consider a few typical optical instruments.

I. Images by Reflection

Reflection corresponds to the special case of $\rho = -1$ as will appear at once from the discussions connected with Fig. 1. This is also the value of ρ_0, in accordance with the definition of this quantity on page 3, and, since the first and second principal points fall together on the surface, as is obvious from the fact that a plane object on the surface of a mirror completely coincides with its image, the equations of Group I are appropriate and convenient. The mirror being plane, $\gamma = 0$ and $c^0_{\lambda+1} = 0$; then the first equation of I shows that C_1 and C' always have opposite signs and are numerically equal; in other words, that image and object are always symmetrically placed with respect to the mirror. The second equation gives the transverse magnification as equal to $+1$, hence the image is erect and equal to the object. When the object is in front of the mirror its image is in the inaccessible region behind it, and for this reason has been called a *virtual* image. This term is useful in our generalized method of treatment provided that we extend it to objects also, in which case we state that a virtual image corresponds to a real object and *vice versa:* thus, the plane mirror of the Newtonian telescope forms a real image of the virtual object behind it.

The interpretation of the third equation is of interest: it shows that the longitudinal magnification is always equal to -1, hence the image of a right-handed helix, however presented to the mirror, is a left-handed helix; so too, the image of a right hand is a left hand. This relation of the image to the object is called *perversion.*

Plane Mirror. — All spherical wave-surfaces incident upon a plane mirror remain spherical after reflection, irrespective of obliquity and of angular magnitude; hence the limitation as regards apertures and proximity of object and image to the axis may be suspended. This is equivalent to saying that the plane mirror is a *perfect* optical instrument, an advantage which it alone possesses. An important conclusion from this fact is that the theory developed in Chapter I is wholly adequate to explain completely the whole of the phenomena presented by such an instrument. The following paragraphs will illustrate this statement.

A plane mirror of infinite extent divides the visible universe into two equal parts, that lying behind the mirror being suppressed and replaced by a perverted image of the half in front. If the mirror is bounded, it may be regarded as a window through which a portion of the perverted universe can be seen by an observer in front of the plane of the mirror, the portion being completely defined by the shape of the window and the position of the observing eye. For example, the smallest mirror in which one can see his whole figure must have half his height and half his width and the observer must be directly in front of it. If one demands complete binocular vision the mirror must be increased in width by one half the interval separating the eyes. As another illustrative example, suppose the problem is to find the shape and size of a plane mirror which will just enable an observer to see the image of a disk the plane of which contains the eye of the observer and which is also parallel to the mirror. From the theorem given above one sees at once that the mirror must be circular and have a diameter one half that of the disk.

The terms "right" and "left" employed in the characterization of images formed by a plane mirror are wholly unsatisfactory since they infer the position of the median plane of the observer; this clearly has nothing to do with the physical phenomena involved. It is true that the terms are properly applied to helices, but only because of a universally accepted convention by no means restricted to purely scientific usage; on the other hand, when we desire to describe the relation of the images of features of a landscape produced by the surface of quiescent water to the corresponding objects, the terms fail, nor can we replace them by terms "inverted," which has a quite different meaning. The true definition is that contained in the equations above, namely perversion ensues whenever M and L have different signs, and then only.

Consider the case of successive reflections in two mirrors. Let *a* 1 and *a* 2, Fig. 7, represent two plane mirrors meeting in *a* at an angle α, the reflecting faces facing each other; also, let 0 be an object in any position between the mirrors. Then i_1 will be the symmetrically placed and perverted image produced by *a* 1. Of this image the mirror *a* 2 will form a perverted image $i_{1,2}$, which is a direct image of *o* changed in azimuth by the angle 2α. Again, the mirror *a* 2 forms a symmetrically placed and perverted image i_2 of which, in turn, the mirror *a* 1 forms an image $i_{2,1}$, which is a direct image of *o* changed in azimuth by 2α in the opposite sense. This construction can be continued with a series of alternate reflections until the image falls behind the mirror. An odd number of reflections yields perverted images, and an even number direct images. If α is an aliquot part

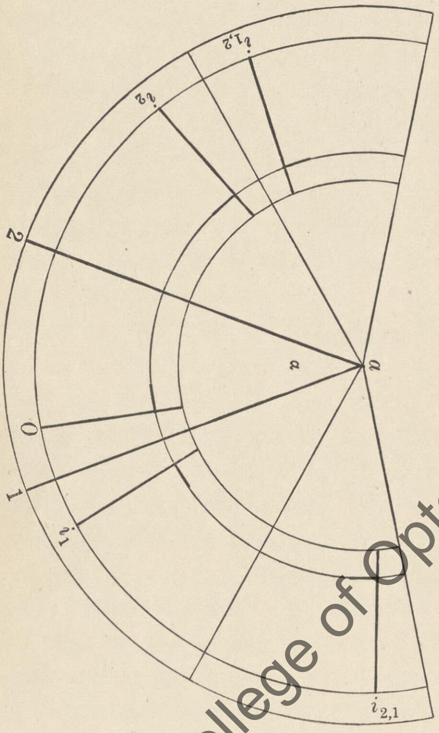

Fig. 7

of 2π the objects included between the mirrors, together with their images, form a self-limited polygonal space; the mirrors in this case constitute the optical system of the kaleidoscope. When α is equal to 90° we

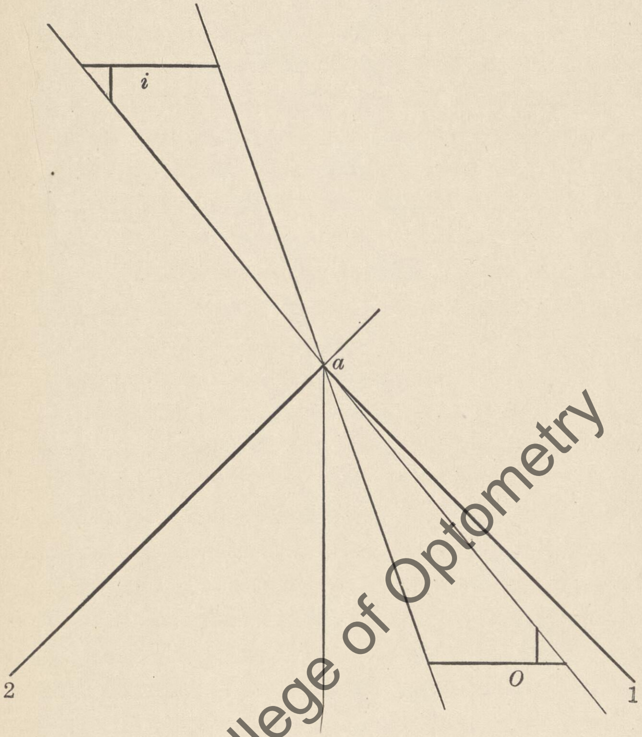

Fig. 8

meet with a particularly interesting solution on account of its extended application in modern practice. Let Fig. 8 illustrate this case. Here $a\,1$ and $a\,2$ are the mirrors with o, the object, placed anywhere between them; the image i which is due to successive reflections taken in either order, is direct, virtual, and exactly 180° from 0 at an equal distance from a. Obviously, object and image may exchange places provided the object becomes virtual and the image real — another way of stating that the reflection is confined to the sides of the mirrors facing each other.

The action of the two mirrors can be described as equivalent to (1) changing the angular position of the object with respect to a by 180°; (2) rotating the object about an axis through its center and parallel to both mirrors by 180°; (3) reversing the direction of propagation of the light. If the light after leaving the mirrors falls upon a second pair whose line of intersection is at right angles to that of the first, the sum of the effects will be a linearly displaced image, completely inverted, with light propagated in the original direction. Porro utilized such a system of four mirrors nearly a century ago to replace the middle member of the terrestrial telescope, not because of greater optical efficiency, but because of a great reduction in length, especially in low powers. For his mirrors he employed, naturally, total reflection from the interior faces of right-angled prisms. This ingenious invention remained practically unused until revived in recent years and applied to binoculars by Zeiss; since that time the use has grown very rapidly and such "prismatic binoculars" are now made by many manufacturers.

If a plane mirror be rotated about a vertical line in its plane, the image of a point in the horizon will revolve about this axis, in the opposite direction, with twice the angular velocity; the image exists only while the object is in front of the mirror. If this image be observed by means of a fixed vertical mirror, the image will seem to move in the opposite direction on account of perversion: if now the second mirror be rotated in the same direction and with the same angular velocity, the image will appear to be at rest. When the two mirrors are rotating as supposed, they retain a constant angle between them and the image of the object, as long as it remains within the angular space embraced by the two mirrors, will be at a constant angular distance from the object. A pair of mirrors with variable and measurable angle can be used to measure the angular separation of two objects provided that one of the objects can be seen directly while the image of the other is, by adjusting the angle between the mirrors, brought to coincidence with it. One commonly used provision for the simultaneous vision of the two is by having the mirrors cover only one-half the pupil. Such an instrument is the sextant in which the freedom from conditions of stability, with accompanying weight, renders the inconvenience attending double vision quite negligible.

Spherical Mirrors. — These offer an especially interesting illustration of the method under discussion because in them all of the ten cardinal points can be immediately found, without calculations, from the definitions alone. Thus, the two principal points and the negative nodal points all fall at the vertex; the two nodal points and the negative principal points correspond at the geometrical center of the mirror, consequently, the two principal foci lie halfway between the vertex and the geometrical center. Since all the cardinal points are known, any one of the groups of equations is as applicable as the others.

There are two cases to be distinguished; first, that in which γ is positive, or the mirror is convex; second, that in which γ is negative, or the mirror is concave. In the first case, $\alpha c^0_{\lambda+1} = 2\,\gamma$, whence, turning to equations of Group I, we see that C_1 has this value for $C' = 0$ and C' takes the same value for $C_1 = 0$, or, in accordance with the statement above, both principal foci lie midway between vertex and geometrical center. For all negative values of C' the values of C_1 are positive; in other words, for all real objects the images are virtual. The second equation shows that these images are erect and always smaller than the object; the third equation, since L is always negative, that the images are perverted.

In the second case, the concave mirror, the value of $\varkappa c^0_{\lambda+1}$ is also equal to $2\,\gamma$, but this is now a negative quantity and the relations of image and object are more complex. Restricting our consideration to real objects, that is, to cases where C' is negative, we see that, as long as $\rho_0 C'$ is numerically less than $2\,\gamma$, C_1 is negative and the images are real; when $\rho_0 C'$ is numerically greater than $2\,\gamma$, C_1 is positive and the images are virtual. In the former group M is negative, and in the latter positive; in all L is negative and all images are, therefore, perverted.

II. IMAGES BY REFRACTION

Plane-parallel Plate in Air.—Let n be the index of refraction, then

$$\gamma = 0 \qquad \rho = \frac{1}{n}$$

$$\gamma' = 0 \quad t_1 = t_1 \qquad \rho_1 = n \quad \rho_0 = 1$$

From equations (a)

$$\begin{aligned} c_1 &= \rho c' & \mu_1 &= (1 - t_1 c_1)^{-1} \\ c_2 &= \rho_1 c'' = \mu_1 c' & &= (1 - \rho t_1 c')^{-1} \end{aligned}$$

Since $\mu_\lambda!$ equals $(\alpha + \beta c')^{-1}$ we have $\alpha = 1$ and $\beta = - \rho t_1$. This is a complete solution of the problem with the first and last vertices as the points of reference; if we desire to substitute the principal points for these and employ the equations of Group I, we have only to calculate the positions of these points according to the method of Section VI of Chap. I. Following this course we find from (j) the expression for ξ' to be indeterminate, but since, from considerations of symmetry, ξ' must be equal to $- \xi$ we have at once

$$\alpha\xi = - \rho_0\xi - \beta, \text{ whence } \xi = \tfrac{1}{2}\, \rho t_1$$

The separation of the principal points is equal to $t_1(1 - \rho)$. Applying the equations of Group I to this case we see that C_1 equals C', and that all three kinds of magnification are equal to unity. The total effect of such a plate may be described as equivalent to transposing the position of the object, real or virtual, by an amount equal to its thickness multiplied by $(1 - \rho)$.

Spherical Magnifier in Air. — In this case it is evident that both nodal points, therefore, both principal points also, fall at the center of the sphere. Plane wave-surfaces passing the center would be refracted at emergence according to (a), so as to have curvature

$$c_2 = \gamma'(1 - \rho_1)$$

the center of which is $\frac{-r}{1 - \rho_1}$ from the second vertex, if r is the radius of the sphere, or $\frac{-\rho_1 r}{1 - \rho_1}$, from the center; this point, however, is the second negative principal point, hence the distance from the center of the sphere to the second principal focus is $\frac{-\rho_1 r}{2(1 - \rho_1)}$, which for crown glass equals $\frac{3}{2} r$, nearly. The fact that a diaphragm, such as might be easily produced by an equatorial groove in such a sphere, appears to make the refractive effect of the sphere the same for points away from the axis as for the axial points, led the opticians of a century ago to regard this type of magnifier as of special excellence, and it appears even to this day on the catalogues of manufacturing opticians under the name of Coddington Lens. In reality the advantage named is wholly specious, for in practice the ocular pupil of the observer is the effective diaphragm; on the other hand, the focal distance is inconveniently small, being only one half the radius, and the cost of construction is materially greater than that of a thin lens of like power.

Plano-spherical Lenses in Air. — To find all the cardinal points apply the operations of (a) with two assumed values of c' according to the following scheme:

$$\begin{array}{ll} c' = 0 & c' = c' \\ c_1 = 0 & c_1 = \rho c' \quad \mu_1 = (1 - \rho t_1 c')^{-1} \\ c_2 = \gamma'(1 - \rho_1) & c_2 = \gamma'(1 - \rho_1) + \mu_1 c' \end{array}$$

whence ρ_0 equals 1, α equals 1, and $\beta = \rho t_1$, and by equations (j) of page 22

$$\xi' = 0 \qquad \xi = \rho t_1.$$

These solutions are perfectly general; we may apply them to the two cases of convex and concave lenses as follows, r being radius of surface.

Convex	*Concave*
$\gamma' = -\frac{1}{r}$	$\gamma' = \frac{1}{r}$
f' lies $-\frac{r}{1-\rho_1}$ from e'	f' lies $\frac{r}{1-\rho}$ from e'
f lies $\frac{r}{1-\rho_1}$ from e	f lies $-\frac{r}{1-\rho_1}$ from e

For crown glass, where n is $\frac{3}{2}$ nearly, we see that the distances separating the principal points from the corresponding principal foci are equal to $2\,r$, and the first principal point is $\frac{2}{3}$ the thickness of the lens from the plane surface towards the second vertex.

Terrestrial Ocular. — In order to exhibit the method as applied to more complicated systems, we may choose a common terrestrial ocular.* The optical constants are given in the following table; all the lenses are plano-convex:

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* This is a type credited to Fraunhofer, and which may well be regarded as a remarkable invention since it has been in universal use for nearly a century. The index of refraction is that of the material employed by me in an actual construction in which the angular field was 45°.

TERRESTRIAL OCULAR

Radii	*Separations*
∞	
− 0.831	0.10
∞	2.56
− 1.050	0.10
+ 1.166	4.10
∞	0.17
+ 0.658	1.85
∞	0.10

$n = 1.514$ for materials of all lenses.

Calculations with these constants were carried through in accordance with equations (a), first with c' equal to zero and second with c' taken as equal to -1.5247. The latter choice should be regarded as purely arbitrary, any other value of c' than zero meeting all theoretical requirements. The following are the results required for our discussion:

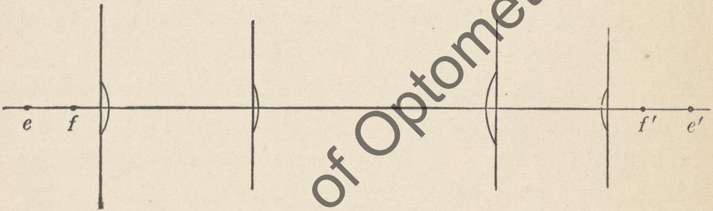

Fig. 9

$$
\begin{aligned}
c' &= 0 \\
c_8 &= 1.5247 \\
\mu_7! &= \alpha^{-1} = -\ 1.2648 \\
\alpha c_8 &= P = -\ 1.2060 \\
\xi' &= 1.485
\end{aligned}
\qquad
\begin{aligned}
c' &= -\ 1.5247 \\
c_8 &= 0.2348 \\
\mu_7! &= (\alpha + \beta c')^{-1} = -\ 0.6705 \\
\beta &= -\ 4.5950 \\
\xi &= \frac{\xi' - \beta}{\alpha} = -\ 1.2967
\end{aligned}
$$

$$\frac{1}{P} = -\ 0.829$$

The interpretation of these results is that the first principal point lies at the left of the first vertex by a distance equal to ξ, then comes the first principal focus 0.829 nearer the vertex; at the other end of the system the second principal focus lies to the right of the eighth vertex by a distance of 0.656, followed by the second principal point 0.829 further. The power, like that of a simple divergent lens, is negative, but it forms real images of distant objects instead of virtual. The principal points are more widely separated than the principal focal points, all four being outside the system.

Human Eye. — The following constants are taken from Helmholtz (Phys. Optik, 2nd ed., p. 140).

r_1	0.7829		n	1.3365
r_2	1.0000	$t_1 = 0.36$	n'	1.4371
r_3	-0.6000	$t_2 = 0.36$	n''	1.3365

$$\rho_0 = \frac{1}{n''} = 0.7482$$

The unit of length is the centimeter, and the index of refraction for the lens is taken as the optical mean of its varying value. From the constants the following quantities are deduced:

$\log \gamma$	.1063		
$\log \gamma'$	.0000	$\log t_1$	9.5563
$\log \gamma''$	.2218 N	$\log t_2$	9.5563
$\log \rho$	9.8740	$\log (1 - \rho)$	9.4011
$\log \rho_1$	9.9685	$\log (1 - \rho_1)$	8.8451
$\log \rho_2$	.0315	$\log (1 - \rho_2)$	8.8757 N

By means of equations (a) together with those on page 22, taking c' equal first to zero and then to $- 0.5$ (a purely arbitrary choice), the following results may be obtained:

$c' = 0$

$\log \gamma$	.1063		
$\log (1 - \rho)$	9.4011		
$\log c_1$	9.5074		
$\log \mu_1$	.0535		
$\log c''$	9.5609	$\log \gamma'$	.0000
$\log \rho_1$	9.9685		8.8451
	9.5294		8.8451
$\log c_2$	9.6110		
$\log \mu_2$	.0691		
$\log c'''$	9.6801	$\log \gamma''$	.2218 N
$\log \rho_2$	.0315		8.8757 N
	9.7116		9.0975
$\log C_3$	9.8061		
$\log \mu_2!$	.1226		
$\log \alpha$	9.8774		
$\log \xi'$	9.7066 N	$\therefore - 0.509$	

	$c' = -0.5$	
log c'	9.6990 N	
log ρ	9.8740	
	9.5730 N	9.5074
log c_1	8.7194 N	
log μ_1	9.9919	
log c''	8.7113 N	
log ρ_1	9.9685	
	8.6798 N	8.8451
log c_2	8.3455	
log μ_2	.0034	
log c'''	8.3489	
log ρ_2	.0315	
	8.3804	9.0975
log C_3	9.1737	
log $\mu_2!$	9.9953	
log β	9.7106 N	
log ξ	9.2459	$\therefore$ 0.176

The second principal point is measured from the third vertex and the first principal point from the first vertex. The constants of Group I are thus completely known, the first equation being

$$C_1 = 0.4828 + 0.7482\,C'$$

When C' is zero, the reciprocal of C_1 is the second principal focal length, and when C_1 is zero the reciprocal of C' is the first principal focal length: these values are 2.071 and $-$ 1.550, respectively. The positions of the nodal points are found either from equation (f) or from their relations to the principal points as stated on page 20. In order to acquire a clear notion of the position of the cardinal points of the normal human eye adjusted for objects at a great distance, the positions referred to the *first* vertex are given in the following table.

f	-1.374	e'	$+0.210$
e	$+0.176$	n'	$+0.732$
n	$+0.698$	f'	$+2.282$

A feature of the human eye of great importance to the theory of vision by means of the optical instruments is the iris. This lies directly upon the anterior surface of the lens and has a variable central opening — the pupil — which, for daylight vision, may be estimated to range from 0.1 to 0.4 of a centimeter.

Cardinal Points of Composite System. — Suppose the principal points and powers of each of two members of

Fig. 10

a composite system are known, as illustrated in Fig. 10, where D is the distance separating the second principal point of the first member from the first principal point of the second; moreover, suppose an object of angular subtense α to be situated at an indefinitely great distance to the left. The image formed by the first member will lie at the distance $\frac{1}{p_1}$ to the right of e_1', and $\frac{1}{p_1} - D$ to the right of e_2, the reciprocal of the latter distance being the value of C' in the general equations of Group I as applied to the second member. The first of the general equations gives

$$\begin{aligned} C_1 = p_2 + C' &= p_2 + p_1(1 - p_1D)^{-1} \\ &= \{p_1 + p_2 - Dp_1p_2\}(1 - p_1D)^{-1} \end{aligned}$$

The reciprocal of this value is the distance from e_2' to the secondary image formed by the second member.

Now consider the dimensions of the two images. The size of the primary image is $\frac{\alpha}{p_1}$, and the magnification of the second member being $\frac{C'}{C_1}$, the size of the secondary image will be $\alpha\{p_1 + p_2 - Dp_1p_2\}^{-1}$ which is obviously the same as $\frac{\alpha}{P}$, if P is the power of the composite system; therefore we have finally the general expression for the power of two combined systems:

$$P = p_1 + p_2 - Dp_1p_2$$

The distance from E', the second principal point of the whole system, to the second principal focus is $\frac{1}{P}$, or measured from e_2' it is $-\frac{Dp_1}{P}$. By a consideration of a reversal of the system we see that the distance from e_1 to the first principal point E is $\frac{Dp_2}{P}$, hence all the cardinal points of the composite system are determined in relation to its components.

In order to illustrate the use of this method of substituting a single set of cardinal points for any number of others, the terrestrial ocular described on page 36 may be chosen. Opticians ordinarily look upon this system as composed of two parts; the first, consisting of the first and second lenses, is called the *erector*, and the second, embracing the third and fourth lenses, is styled the eye-piece. In the following table the first column contains the principal points and powers of each of the lenses; the second column contains the powers and principal points of erector and eye-piece, respectively, while the third column contains the power and principal points of a simple system which replaces the two composite systems of the second column.

Single Lenses		*Erector* / *Eye-pieces*		*Total System*	
e_1	0.0661	D_1	2.6261	D	12.096
e'_1	0.0000	P_1	.3130	P	− 1.208
p_1	0.6186	E_1	4.1079	E	− 5.467
e_2	0.0661	E'_1	− 5.1902	E'	3.133
e'_2	0.0000				
p_2	0.4896				
e_3	0.0000	D_2	1.9623		
e'_3	− 0.1123	P_2	.5463		
p_3	0.4409	E_2	2.8062		
e_4	0.0000	E'_2	− 1.5838		
e'_4	− 0.0661				
p_4	0.7812				

In the first column the two points of reference are, in each case, the first and second vertices of the lenses; in the second column they are the first principal point of the first lens and the second principal point of the following lens; and for the last column the two points are E_1 and E'_2, that is, the first principal point of the first composite system and the second principal point of the second composite system. The places of E and of E', referred to the first and last vertices of the system, respectively, are given by the equations

$$\begin{aligned} e_1 + E_1 + E &= -\ 1.293 \\ e'_4 + E'_2 + E' &= \quad 1.483 \end{aligned}$$

These values, with that of P, agree with those on page 37 derived by direct calculations.

From the expression for the combined power of two lens systems on page 41 one sees that the power of the terrestrial ocular, and therefore the magnification of the telescope of which it forms a part, can be greatly varied by a change of the distance separating the eyepiece from the erector. Such a variable system has long been in use under the name of "panaratic eyepiece"; its use, however, is to be recommended only in a very qualified manner, since it has of necessity a more restricted field and thus loses a property generally far more valuable than numerical magnification. As a matter of fact, the current practice is to make magnifications of spy-glasses quite too high, probably on account of the less cost of such instruments since they demand smaller lenses.

Telescope: Microscope. — A knowledge of the cardinal points of the telescope, or of the microscope, is quite inadequate for a discussion of their office as aids to vision. This arises from the fact that small changes in adjustment, such as are constantly made to suit varying demands in the observer, are often accompanied by enormous changes in the positions of the cardinal points without appreciable changes in the action of the instrument. It is highly desirable, in these important cases at least, to establish a different method of treatment involving only elements which are invariable as long as the instrument remains practically the same to its user.

The two instruments are treated together because there is, as will at once appear, no logical distinction. Suppose an optical aid to normal vision be adjusted upon a distant object: the instrument is called a telescope. Now imagine the object to approach the observer continuously, the instrument undergoing simultaneous modification so that the object always appears distinct; as long as the object is remote the name of the instrument remains unchanged, but when the object has attained a position close to its forward end, the instrument is called a microscope. Formerly there was no practical difficulty in distinguishing the two classes of instruments since, in general, such optical instruments were used either for distant objects or for those quite near; but with the modern wide growth of the demands upon the optician the essential continuity of the two must be recognized by one who wishes to treat their elementary theory in a logical manner.

Let Fig. 11 represent any optical apparatus whatever used as an aid in seeing o'. Let a be an object immediately in front of the system whose image is a'; if a' is close to the back of the system its position is the most favorable position for the pupil of the eye, for then the oblique, as well as the direct, wave-surfaces pass through a'. Designate the distance from the eye, thus placed, to the object by D, and the distance from a to the object by d, as represented in the figure. If d is large the instrument is a form of telescope, if small it would be called a microscope, as is explained above.

As a rule the apparent angular magnitude of o' will be modified by the instrument: indeed, more often than not the immediate end sought in the use of the instrument is an increase of the apparent size. Since the observer cares ordinarily little as to the exact position of the image under inspection provided that it is quite distinct, the measure of angular magnification is the quantity of particular interest. This is given at once by the general equation (i), page 13, namely

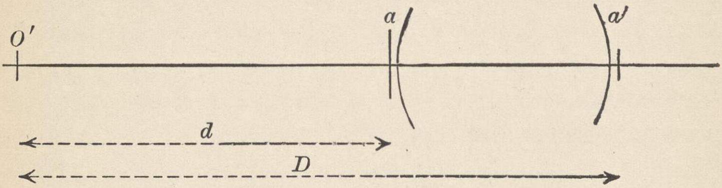

Fig. 11

$$N = \rho_0 g$$

where g is the ratio $\frac{a}{a'}$.

If the instrument is a telescope the first and last media are alike and ρ_0 therefore equals unity. Where a is the whole diameter of the objective it is only necessary to measure this and the diameter of its ocular image to find the magnifying power; that is to say, the ratio of the apparent dimension of the image to that of the object. This method of determining the magnification of a telescope has long been in use and is commonly attributed to Ramsden.

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The Ultimate Power of the Telescope — of the Microscope. — In any instrument employed as an aid to vision the ultimate power depends on the natural limitations of the eye and on the finite length of light waves. Experiments, which are entirely in keeping with the anatomical structure of the retina as shown by Helmholtz, demonstrate that the resolving power of an exceptionally acute human eye is limited to one minute of arc. Thus, if we have a series of parallel black lines on a white ground of which the angular separation is that named, *e.g.* 344 lines to the inch at ten inches from the eye, under a suitable illumination the surface will appear finely lined instead of a uniform gray as under less favorable conditions. It will be found that for this critical vision the pupil will have an opening not far from one fifteenth of an inch; if the pupil is much larger than this the considerable spherical and chromatic aberrations inherent in the eye will reduce the resolving power, and, if very much smaller, the diffraction effect due to the length of the waves of light to which the retina responds, will also produce a like limitation. We may deduce a general rule from this principle that the eye-circle should not exceed one fifteenth of an inch in diameter nor be usefully less than one half that amount when the conditions of illumination are at their best. In the microscope the illumination is quite at command and the theoretical rule may be accepted without limitation. In telescopic vision, where the illumination is often far below that most favored, the lower limit of magnification is far too high and the eye-circle should have two, or in case of nocturnal vision or in observing faint astronomical objects, more than three times the value stated. The higher limit, corresponding to an eye-circle of one thirtieth of an inch, can hardly be surpassed with real advantage to the observer.

The rule admits of the simplest possible application to the telescope since, in accordance with the statement above, the magnification attending the larger eye-circle is fifteen times the diameter of the objective in inches, and hence the resolving power of a perfect telescope of which the objective has a diameter of one inch is fifteen times that of the unassisted eye and larger telescopes have greater resolving power in direct ratio to their augmented apertures. This theoretical conclusion, which implies that an observer should be able with a perfect four-inch telescope to just divide a double star of 1-inch separation, is in complete accord with experience.

It is worth noting that the focal length of the telescope does not enter at all in the discussion; moreover, that there is no limit, other than that set by mechanical difficulties, to the possible power of the telescope. It is true that the difficulty in securing larger disks of optical glass than those now found in our greater telescopes and, possibly, a limit fixed by elastic yielding in lenses of greater dimensions, suggest that we may have already reached the end.

The theory of the microscope is somewhat less simple because of certain restrictions of convenience of which the most notable one is that of the distance between the ocular and the anterior system. This distance is now very generally fixed by microscope makers at 16 cm. or 6.3 inches. Moreover, since the eye-circle must have a diameter not greater than one fifteenth of an inch for the maximum resolution, the power of the ocular is determined by the effective diameter of the back of the objective, a dimension which is obviously arbitrary. In order to make the problem definite let it be assumed that this diameter is 0.3 inch, which is quite common in practice; then the first equation of Group I on page 15 shows that the power of the ocular must be 0.5.

The magnification of the objective, as given on page 9, is

$$M = \rho_0 \frac{\sin \omega'}{\sin \omega_{\lambda+1}},$$

where, as is immediately seen from Fig. 11,

$$\omega' = - tg^{-1} \frac{a}{2\,d'},$$

and as determined by the adopted conventions

$$\sin \omega_{\lambda+1} = \frac{.15}{6.3} = .0238;$$

substituting these values in the equation for magnification

$$M = - 42\, \rho_0 \sin \omega'.$$

The largest possible value of the sine is unity and the corresponding value of M is $- 42\, \rho_0$, the negative sign indicating an inversion of the image. The first and second equations of Group I show that the power of the objective must have a minimum value of $6.83\, \rho_0$, and as the magnification of the ocular referred to the standard distance of 10 inches is equal to 5, the minimum magnification necessary to the highest degree of resolution in microscopic vision is 210 ρ_0. The general conclusion is that magnification from 200 to 400 in "dry" objectives in which ρ_0 is equal to one, and from 300 to 600 for homogeneous immersion systems, exhaust the useful powers as far as study of structures is concerned, a conclusion quite in accord with the experience of microscopists. Professor Abbe's admirable invention of the apochromatic objective does not modify the preceding conclusions, as the argument tacitly assumed perfect correction in all the elements of the instrument.

CHAPTER III

Color Errors and Corrections

Whenever a wave-surface is changed by refraction, excepting the case in which ρ equals minus unity or that of reflection, the change with composite light varies with the wave-length. A more convenient statement for our purpose is, that the changes vary with refrangibility of the light in question, the refrangibility being defined by the index of refraction for that particular wave length in some substances arbitrarily chosen as a standard. If all of the values of ρ which enter our calculations are known functions of this index of refraction, designated by n, it is easy to find the variation in value of any c in equations (a) for light of any prescribed refrangibility by differentiating these equations with respect to n. Such equations, derived from (a), are the following:

$$\begin{aligned}
\frac{dc_1}{dn} &= (c' - \gamma)\,\frac{d\rho}{dn} \\
\frac{dc_2}{dn} &= (c'' - \gamma')\,\frac{d\rho_1}{dn} + \mu_1^2\rho_1\,\frac{dc_1}{dn} \\
\frac{dc_3}{dn} &= (c''' - \gamma'')\,\frac{d\rho_2}{dn} + \mu_2^2\rho_2\,\frac{dc_2}{dn} \qquad \text{(k)} \\
&\quad \cdots\cdots\cdots \\
\frac{dc_{\lambda+1}}{dn} &= (c^{\lambda+1} - \gamma^{\lambda})\,\frac{d\rho_\lambda}{dn} + \mu_\lambda^2\rho_\lambda\,\frac{dc_\lambda}{dn}
\end{aligned}$$

Having obtained the last of the differential coefficients by the successive computations we have as the difference of focal distances for light of two refrangibilities differing by δn

$$-\frac{1}{C_{\lambda+1}^2}\frac{dc_{\lambda+1}}{dn}\delta n$$

As an application of the method here developed we may discuss the color error of the human eye, the cardinal points of which have already been derived, although it is necessary to add to the constants given above a knowledge of the dispersive power of the media.

Color Error of the Human Eye.—For the purpose the following table of refractive indices is quite sufficient.

	Aqueous humor	*Lens*	*Vitreous humor*
	$n_0 = 1.3365$	$n_0' = 1.4371$	$n_0'' = 1.3365$
Spectral line	$n - n_0$	$n' - n_0'$	$n'' - n_0''$
A	− .00558	− .00876	− .00558
C	− .00370	− .00581	− .00370
G	+ .00590	+ .00926	+ .00590
H	+ .00900	+ .01413	+ .00900

The dispersion of the aqueous humor and of the viteous humor eye are taken directly from Helmholtz, but the dispersion of the lens is higher than that adopted by him.* Since the choice of the medium which shall serve to characterize the light is purely arbitrary, that of the aqueous humor is obviously as convenient as any other. Distinguishing the indices of refraction of the second and third media by n' and n'', respectively, the following values are readily deduced.

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* See C. S. Hastings, Amer. Jour. Sci., vol. xix, p. 205, for more extended treatment of this problem.

$$\frac{dn'}{dn} = 1.57;\ \frac{dn''}{dn} = 1.00;\ \frac{d\rho}{dn} = -\,0.560;\ \frac{d\rho_1}{dn} = -\,0.326;\ \frac{d\rho_2}{dn} = 0.374$$

With these values supplementing those found on page 37 of the preceding chapter the color errors can be calculated from equations (k) as follows:

$$\begin{array}{llll}
\log (c' - \gamma) & = \ .1063\,N & & \\
\log (c'' - \gamma') & = 9.8036\,N & \log \mu_1 & = .0535 \\
\log (c''' - \gamma'') & = \ .3415 & \log \mu_2 & = .0691
\end{array}$$

$$\begin{array}{lrlr}
\log (c' - \gamma) & .1063\,N & & \\
\log \frac{d\rho}{dn} & 9.7482\,N & & \\
\cline{2-2}
\log \frac{dc_1}{dn} & 9.8545 & & \\
\log \rho_1 & 9.9685 & \log (c'' - \gamma') & 9.8036\,N \\
\log \mu_1^2 & 1070 & \log \frac{d\rho_1}{dn} & 9.5132\,N \\
\cline{2-2}\cline{4-4}
 & 9.9300 & & 9.3168 \\
\cline{2-4}
\log \frac{dc_2}{dn} & 0.0247 & & \\
\log \rho_2 & .0318 & \log (c''' - \gamma'') & .3415 \\
\log \mu_2^2 & .1382 & \log \frac{d\rho_2}{dn} & 9.5729 \\
\cline{2-2}\cline{4-4}
 & 0.1944 & & 9.9144 \\
\cline{2-4}
\log \frac{dc_3}{dn} & 0.3776 & &
\end{array}$$

This result is most readily expressed in terms of ordinary experience by the following considerations:

From the transposition equation (d), page 10, may be derived

$$\frac{dC_1}{dc_3} = (1 - \xi' c_3)^{-2}$$

whence, in this case, since $\log(1 - \xi' c_3)$ is equal to .1224,

$$\log \frac{dC_1}{dn} = .1328$$

For the total variation from extreme red to extreme violet for which $\delta n = .01458$ as appears from table above,

$$\delta C_1 \text{ equals } 0.0198.$$

From the general equation above for the eye, page 39,

$$\delta C_1 = 0.7482\, \delta C',$$

in which equation $\delta C'$ is the measure of accommodation, its maximum value being Donders's Constant of accommodation. The change of accommodation requisite for maintaining sharp vision from one end to the other of the spectrum is .0265, the centimeter being the unit of length. As this is about the limit of accommodation for the human eye in mid-life it is surprising that its large color error is so generally overlooked.

Color Corrections. — The variety of possible corrections, or partial corrections, of errors due to varying refrangibility is so great that it is remarkable that only a single term to describe them has been evolved. For the general discussion of the theory of optical instruments a greater definiteness in language is imperative, hence I shall venture to employ three new terms after specifying the character of the corrections corresponding to each.

The first correction of color defect in optical images, by far the earliest known, is the simplest, since it can be accomplished by the use of a single material. In it the power of the system is independent of the refrangibility of light, although the positions of the cardinal points are not so. As the images are, in such a system, distributed in the order of wave-length, it is quite illogical to call such a system achromatic; such a system may be, in accordance with its characteristic property, called *isodynamic*.

With two media whose dispersive powers are unequal an achromatic system may be constructed by compensation, one portion having an effect opposed to the other. At present, however, no two glasses are known which will admit of perfect compensation for the whole range of spectral colors at once; for example, a binary system corrected as perfectly as possible for visual images would not at the same time yield a satisfactory correction for photography. The nature of the correction in a given case may be completely defined by the color, or wave-length, for which it is most perfectly adapted. Thus, a system corrected for spectral yellow would be satisfactory for visual purposes while a correction for blue would be better adapted to photography, and an intermediate type corrected for green would meet the requirements of an ordinary camera where a difference of the adjustment for visual and photographic purposes is undesirable. Such systems are called achromatic, with a meaning perfectly established by custom and familiar in practice.

It is, however, possible to construct an optical system in which the focal points for all refrangibility fall together with far greater accuracy than in the binary achromatic, although the principal points are widely distributed. As this correction offers certain advantages in some cases — as will be shown later — it will be convenient to give it the name of *orthokumatic* system.

Although binary achromatics, with the media at present within the command of the optician, are imperfect, as explained above, it is possible to secure very perfect correction by a proper choice of not less than three unlike media. If such a system is properly constructed the principal points and focal points, respectively, will correspond for all colors. Such a perfect system may be styled *isokumatic.* It is obvious that a system of this type is also orthokumatic and achromatic at the same time.

There is another error in wide-angle optical systems which betrays itself as a color error, the meaning of which was first made obvious to opticians by Professor Abbe, who devised his apochromatic system to correct it. As this is outside the scope of the present chapter, which is confined to the consideration of small apertures, its consideration will be deferred.

With the above definitions we may turn to a discussion of the means of securing the various corrections named.

Isodynamic System. — The power of a system composed of two elements is, as shown on page 41,

$$P = p_1 + p_2 - Dp_1p_2.$$

The derivative of this with respect to the refrangibility of the light, set equal to zero, takes the form

$$\frac{dP}{dn} = 0 = (1 - Dp_2)\,\frac{dp_1}{dn} + (1 - Dp_1)\,\frac{dp_2}{dn}.$$

For simplicity, suppose that the elements are made of the same material and that p_2 equal ap_1, then the above equation gives

$$0 = 1 + a - 2\,Dap_1$$

or

$$2\,D = \frac{1}{p_1} + \frac{1}{p_2}.$$

It follows from this proposition that two single positive lenses of similar glass separated by an interval of one half the sum of their focal lengths between the proximate principal points will have unvarying magnification for all colors. Such is the construction of the familiar Huyghenian ocular which consists of two plano-convex lenses, the convex sides being turned towards the incident light and the dimensions being fixed by the ratio:

$$f_1 : D : f_2 = 1 : 2 : 3.$$

This ocular — sometimes called the negative eye-piece — is really a very remarkable invention, which statement is readily admitted when one recognizes that even the resources of an art, now developed for more than two centuries after its origin, are incapable of designing an ocular more perfectly adapted for the use with the seventeenth century telescope. The fact that so accomplished an optician as Airy quite failed in his efforts to improve upon the Huyghenian construction is instructive also, but the superior excellence of the ocular for the primitive telescope of great length does not by any means exist for the modern telescope; it is, on the contrary, quite imperfect and it is difficult to understand, except perhaps on account of its simplicity and cheapness, why it is so universally used.*

Achromatic System. — For the sake of simplicity in the development of the theory of this color correction, we shall assume the case of a doublet with a focal length of 100 and of which the thicknesses and separation are negligibly small. In practice, of course, our solution must be regarded as only a first approximation, although often very close, which may be amended by use of the exact formulas (a) and (k).

Let n and n' be, respectively, the indices of refraction of the materials of the first and of the second lens: also let $\gamma - \gamma'$ equal A and $\gamma'' - \gamma'''$ equal B. From (a) with the substituted symbols —

$$C = A(n - 1) + B(n' - 1).$$

The condition of achromatism is that this value of C shall be invariable with any change in refrangibility of light, thus:

$$\frac{dC}{dn} = 0 = A + B\frac{dn'}{dn}.$$

In order to interpret this equation let us suppose that the materials employed are ordinary crown glass and dense flint glass, such as those in the Washington 26-inch objective discussed in Chapter VI, types which are very commonly used in modern telescopes. In a certain pair of such materials which I had occasion to employ I found —

$$n' - n'_F = 1.9710(n - n_F) + 9.226(n - n_F)^2.$$

* Further analysis with descriptions of improved types of oculars may be found in a paper entitled "Astronomical Oculars," in *Popular Astronomy*, June-July, 1926, p. 375. See also Chap. VI below.

The derivative of this equation yields —.

$$\frac{dn'}{dn} = 1.9710 + 18.452(n - n_F).$$

It will be observed that the differential coefficient varies continuously with n and consequently, for a determinate solution, it is necessary to fix upon the best value for n in the problem at hand. For a visual telescope the value corresponding to the Fraunhofer line λ 5616 gives the best color correction. Let us designate these particular values of the indices by n_0 and n'_0; then, by substituting the value of n_F in terms of n_0 in the foregoing equations, we find

$$n' - n'_0 = 1.8800(n - n_0) + 9.226(n - n_0)^2.$$

$$\left(\frac{dn'}{dn}\right)_0 = 1.8800$$

The first two of the equations of this section now become

$$\begin{aligned} .0100 &= A(.52016) + B(.62026) \\ .0000 &= A \qquad\qquad\quad + B(1.8800) \end{aligned}$$

of which the solution is

$$A = .05258; \ B = -\ .02796$$

There is nothing here to fix the ratios of γ to γ' and of γ'' to γ''', hence we may impose two arbitrary conditions in addition, and must do so before a definitive solution of the question of construction is attained.

One of these conditions would, in almost all cases in practice, be that the spherical aberration of the system shall vanish; the second condition is less important. It may be, for example, that the extent of field shall be as large as possible; that γ'' shall equal γ' so that the two lenses can be cemented together, or, a plan often followed by telescope makers who have an inadequate command of theory, that γ shall equal $-\gamma'$, since this, making the first lens equi-convex, somewhat lessens the cost of construction.

The foregoing method of determining the proper conditions for best color correction is perfectly adapted to determining the residual errors after meeting conditions of achromatism, or in other words, the secondary color error of the system. To accomplish this, let C_0 be the value just calculated when n equals n_0, and C the value when n_0 is increased by δn. Since δn is always a small quantity we may write:

$$C = C_0 + a_1\delta n + a_2\delta n^2 + \text{etc.}$$

This equation is equivalent to

$$\delta C = \left(\frac{dC}{dn}\right)_0 \delta n + \frac{1}{2}\left(\frac{d^2C}{dn^2}\right)_0 \delta n^2 + \text{etc.}$$

Now A and B are purposely chosen so that $\left(\frac{dC}{dn}\right)_0$ equals zero, hence, since $\frac{1}{2}\left(\frac{d^2C}{dn^2}\right)_0$ is equal to $9.226\,B$, we have

$$\delta C = 9.226\,B\delta n^2$$

This equation shows that C_0, since B is negative, is a maximum. The corresponding changes in the focal length are, clearly,

$$\therefore \delta F = -\frac{\delta C}{C_0^2} = 2580\,\delta n^2$$

This is the secondary color error of this achromatic and it would betray itself in practice in a phenomenon which the working optician calls "outstanding color." To the users of large telescopes the error is customarily recorded in a so-called *color curve* which is found by a spectroscopic study of focal images. The method here given is far easier and more accurate. Thus, in the example chosen, we have the following values for the five refrangibilities catalogued.

δn	$-.003987$	$.000000$	$+.004936$	$+.010395$	$+.013083$
δF	$+.041$	$.000$	$+.063$	$+.279$	$+.442$

The importance of this secondary chromatic error is quite undervalued. It is by far the most serious defect of our modern telescopes and it necessitates a length of the telescope which must increase more rapidly than the increase in aperture, if one demands full optical efficiency. For example, a ratio of length to aperture for a 12-inch objective is ordinarily taken as equal to 15 for a standard construction, but if the aperture and power of the instrument are to be increased at a like rate it is found necessary to materially increase this ratio. The reasons for this are readily deduced from the discussion above. As a matter of fact it may be positively asserted that every large equatorial in existence could be appreciably increased in optical power by an increase in length, or, to state the fact in another way, all great equatorial telescopes of the present day are too short. The fact that the present practice, in view of the greater expense necessary to mount and protect longer telescopes, undoubtedly yields the highest general efficiency only emphasizes the importance of eliminating this secondary error. Unfortunately, the accumulated experience of telescope makers since Fraunhofer has demonstrated that no hope of notable improvement is to be looked for in varieties of the familiar silicate glasses. When the new materials which the researches of Otto Schott and his associates, supported by the enlightened policy of the Prussian government, were rendered serviceable to the art of optics, there were doubtless many who hoped for a great advance in this particular direction. Unfortunately, this hope has as yet failed of attainment, although forty years have elapsed since the first catalogue of the Jena glass makers was issued. The most marked innovation in the materials came from a substitution of phosphoric acid or of boracic acid for the silicic acid of the older types. Neither of these new kinds is free from serious defects from the constructor's standpoint. The phosphate glasses generally lack permanence and are apt to corrode when exposed to the air, while the borate glasses have a yellow color which becomes a serious defect in thick pieces. Of those which I have tried myself may be cited the binary combinations of potash-silicate-crown, No. 339 (type of 0.13 of the Jena Catalogue) with the borosilicate-flint 0.161 of which I have made a number of telescope objectives. As this is probably the best binary combination yet attainable, it is worthy of detailed examination. The optical constants of the two glasses, determined with care, are given in Appendix B. These indices, although derived from careful measurements, are not to be regarded as accurate beyond the fifth place of decimals on account of the small size of the prisms at command. Notwithstanding this fact the relations of the two are expressed with quite sufficient accuracy by the formula

$$n'-n'_0=-7(10^{-6})+[.13098](n-n_0)+[9.9565](n-n_0)^2,$$

where n' and n refer to the indices of the flint and of the crown, respectively, as in the previous discussion, and the numbers in the brackets are logarithms of required coefficients. If the residuals derived from the calculated and observed values of the indices are assumed to be due to errors of observation alone, it will be found that the indicated probable error of any one measured value is $\pm 9(10^{-6})$, and since greater accuracy could hardly be claimed for the measurements we are led to conclude that not only is the formula adequate but also that a systematic error, should such exist, can hardly be found in the sixth place of decimals.

We find by substituting the constants thus acquired in the three equations,

$$\begin{aligned} C &= A(n-1) + B(n'-1) \\ \frac{dC}{dn} &= A \qquad\qquad + B\frac{dn'}{dn} \\ \frac{d^2C}{dn^2} &= \qquad\qquad\qquad B\,\frac{d^2n'}{dn^2}, \end{aligned}$$

for C_0 equal to 0.01 yield,

$$A = .096872,\quad B = -\,.071650,\quad \delta F = .648\;\delta n^2$$

The last term becomes + 0.114 for light of wavelength corresponding to the Fraunhofer line h, an error practically only one fourth the magnitude of that found in ordinary telescopes. I have made a considerable number of telescopes involving the use of these materials of which the most interesting was the largest permitted by the disks at command, namely, one of

2.75 inches aperture and 37 inches focal length. The secondary color, only one sixteenth as conspicuous as that of the ordinary construction, was, as might be anticipated, quite imperceptible and in objects, such as the details of sun spots and the *C* ring of Saturn where feebly luminous surfaces are closely contiguous to bright surfaces, the optical efficiency was quite equal to that of the ordinary construction fifty per cent larger in diameter. It is much to be regretted that the crown glass becomes gradually soiled in contact with the atmosphere so that it requires too frequent cleaning, otherwise I know no two glasses which compare in excellence with those under consideration.

The Jena glass makers have more recently invented two glasses which they name telescope crown and telescope flint; these are recommended as giving achromatic constructions notably superior in respect to color correction as compared to the ordinary types in use. Although I have not investigated these glasses personally, there is enough concerning them already published to enable us to fairly estimate their value. There are three sources from which we may derive the required data, namely, (a) The Zeitsch. f. Instrumentenkunde, 19, page 177, where Dr. R. Steinheil gives his determination of the indices of refraction for the two glasses; (b) the Schott und Genossen, catalogue of 1902; and, (c) a description by Dr. Max Wolf of a telescope made of these materials in the same volume cited above, page 1. From the first of these sources I find the equation:

$$n' - n'_E = -23(10^{-6}) + [.07251](n - n_E) + [.2041](n - n_E)^2,$$

which represents the recorded values of the indices perfectly if a probable error of ± $16(10^{-6})$ may be accorded to each. The measured indices for Fraunhofer E are 1.529283 and 1.525187, respectively, that for the crown being the greater. With these constants I have calculated the color error for an achromatic doublet of a focal length of 100 inches and reproduced it in Fig. 12 below. In the same figure I have entered the color errors as calculated, first, under the assumption that Steinheil's measures are without error, and second,

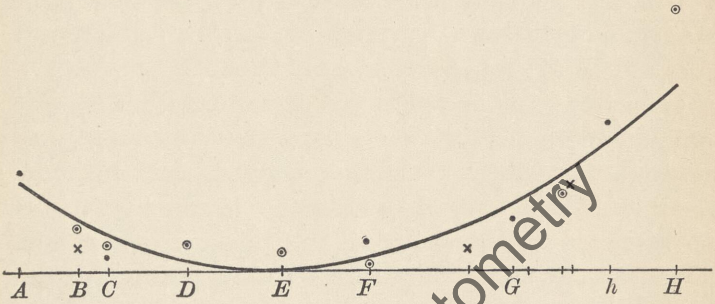

Fig. 12

from the Jena Catalogue values assumed to be without error, and finally, from Dr. Wolf's measures reduced to the same scale. In this figure the abcissas, measured from E as an origin, are proportional to the values of $n - n_E$ while the ordinates are equal to the increment in focal length in changing from n_E to any n. Inspection of the figure will show that the theoretical curve, drawn in full line, is far more likely to express closely the real properties of such a doublet than any one of the three sets of differences. The direct interpretation of Dr. Wolf's measures is not quite easy. They were, it is true, extremely difficult to make since the maximum deviation of light in his instrument was less than one and one half degrees, but I have entered in the figure the points marked in his diagram. The constants of the solution from which the curve of Fig. 12 is constructed are

$$\log A = 9.0545 \qquad \delta F = 1534\,\delta n^2$$
$$\log B = 8.9820\,N$$

The secondary color error is therefore 63 per cent of that of the standard construction while the deep curvatures of the lens surfaces require a ratio of focal length to aperture much greater than the customary ratio with ordinary types of telescope glasses. It shows, however, of the secondary color — taking into account the relative increase of the focal length — a reduction to one third of that of the ordinary achromatic. That this brings with it a material increase in optical power is abundantly shown by Dr. Wolf's description of the tests made with his telescope.

Orthokumatic Correction. — Let there be two systems of indefinitely thin lenses separated by the distance t; further, let C_1 be the curvature of axial wave-surfaces after refraction at the first system, which becomes C'' at incidence on the second system, then we may write:

$$C'' = \mu C_1 = (1 - C_1 t)^{-1} C_1$$

$$\frac{dC''}{dC_1} = \mu^2$$

$$\frac{d^2C''}{dC_1^2} = 2\,t\mu^3$$

etc., etc.

If C_1 is increased by the amount δC_1, the increment of C'' is given by the equation

$$\delta C'' = \mu^2 \delta C_1 + t\mu^3 \delta C_1^2 + \text{etc.}$$

When, as a special case, the first system is a single lens the following equations hold:

$$C_1 = A(n-1), \quad \delta C_1 = A\delta n$$

whence

$$\delta C'' = \mu^2 A\delta n + t\mu^3 A^2 \delta n^2 + \text{etc.}$$

This series converges rapidly unless μ is very large, and the coefficient of δn^2 is positive whenever μ has a positive value. Both these conditions are assured when the separation of the system is considerably less than the focal length of the first, while the second condition suggests a possibility of correcting chromatic aberration to the second power of δn by means of a binary in which, as with all known glasses, the coefficient of δn^2 is negative. A system in which this correction is thus attained is orthokumatic as defined above.

As an example illustrative of the method, suppose it be required to design an orthokumatic of materials Ordinary Crown with Crown 339 and Flint 0.161, the characteristics of which are given in Appendix B; moreover, for the sake of simplicity without material loss of generality, it will be assumed that the first lens is a crown lens of power 0.01 for n_F, that the second system is a binary of power zero for light of the same refrangibility, and, finally, that the thicknesses of all these lenses are negligibly small. Representing the curvature sums for the three lenses by A, B, and C, respectively, and the indices of refraction by n, n', and n'', the equations of condition are as follows:

$$p_1 = .0100 = A(n_F - 1);\; p_2 = .0000 = B(n'_F - 1) + c(n''_F - 1),$$

$$C_2 = B(n' - 1) + c(n'' - 1) + C'',$$

$$\frac{dC_2}{dn} = 0 = B\frac{dn'}{dn} + c\,\frac{dn''}{dn} + \frac{dC''}{dn},$$

$$\frac{d^2C_2}{dn^2} = 0 = B\frac{d^2n'}{dn^2} + c\,\frac{d^2n''}{dn^2} + \frac{d^2C''}{dn^2}.$$

The solution of these equations yields:

$$A = .019040,\; B = 1.0360,\; C = -\,.9518,\; t = 73.$$

The above values demand such moderate curvatures of the lens surfaces in respect to their apertures that the practical construction is easy, but there is an unnoted defect in the images produced by such a system, namely, a chromatic difference of magnification. This is due to the fact that, although the second principal foci for all colors fall together at the same point, this condition is far from true for the second principal points. The positions of the latter can readily be deduced in the following manner from equation (j) which becomes for the case in hand

$$\xi' = \frac{\alpha - 1}{\alpha c^0_{\lambda+1}} = \frac{1 - \mu}{C^0_2}.$$

From this we deduce

$$\frac{d\xi'}{dn} = -\,\frac{1}{C^0_2}\,At\mu^2.$$

This expression multiplied by δn gives the distribution of the second principal points and shows that they lie from 4.59 to the right of the first vertex for red light of refrangibility C to 4.49 to the left for violet light h. There is thus the excessive total chromatic difference of magnification of 9 per cent, which would render such a system objectionable in an ordinary telescope. It is possible, however, to reduce this defect by a skillful choice of materials.*

Isokumatic Correction. — If we have three thin lenses of three different materials in contact, we may express the power of the system by this equation:

$$C_3 = A(n - 1) + B(n' - 1) + c(n'' - 1),$$

whence

$$\begin{aligned} \frac{dC_3}{dn} &= A + B\frac{dn'}{dn} + C\frac{dn''}{dn}, \\ \frac{d^2C_3}{dn^2} &= B\frac{d^2n'}{dn^2} + C\frac{d^2n''}{dn^2}. \end{aligned}$$

These three equations will enable us to find values for A, B, and C which will cause the derivatives to vanish, thus yielding perfect color correction. Such a construction meets the definition given above for an isokumatic.

If the constructor is confined to the ordinary types of silicate glasses the numerical values of the curvature constants are so great that the method can only be employed in cases where the focal length of the system is very great compared to the aperture — cases in which binary achromatics are entirely adequate; but if borate or borosilicate glasses are available the case is different. As an illustration of the method we may calculate the curvature sums for a system made of dense flint 1237, crown 339 and flint 0.161 as defined in Appendix B with a power of 0.0100. We may use n, n' and n'' to denote the indices of refraction of the materials in the order named, then, since

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* A description by the writer of an orthokumatic telescope can be found in the Journal of the Optical Society of America, vol. II, p. 63.

$$(C_3)_F = A(.62977) + B(.53007) + C(.57710)$$

$$\frac{dc_3}{dn} = A(1.9710) + B(1.0140) + C(1.3782)$$

$$\frac{d^2c_3}{dn_2} = 2\,A(9.226) + 2\,B(0.3144) + 2\,C(1.7660)$$

the equations above yield, when the first is set equal to 0.0100 and the others each equal to zero,

$$A = .01915,\ B = .13025,\ C = -\ .12322.$$

These values imply radii of curvatures for the system which renders it quite possible to make a telescope objective of the ordinary ratio of focal length to aperture without introducing difficulties in elimination of spherical aberration.*

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* For many years I have used an isochromatic telescope with an orthokumatic collimator on my spectrometer with a convenience which leaves nothing to be desired. Only the ocular is movable so as to adjust for chromatic aberration of the eye, although even this adjustment might be eliminated by a proper correction in the ocular itself.

CHAPTER IV

OBLIQUE REFRACTION AT A SPHERICAL SURFACE — SPHERICAL ABERRATION — ZONAL DIFFERENCE OF MAGNIFICATION

When a spherical wave-surface is incident at an oblique angle on a refracting surface, the refracted wave-surface will not in general remain spherical, but it is clear that the maximum and minimum curvatures must lie in the plane of symmetry and in the plane of right angles to this plane. The plane of symmetry shall be called the meridional plane and the curvature of the sections of wave-surfaces in this plane shall be designated by $\underline{c}'$, $\underline{c}_1$, etc. The curvatures of sections at right angles to this plane shall be called zonal sections and the corresponding curvatures are designated by $\overline{\underline{c}}'$, $\overline{\underline{c}}_1$, etc. We proceed first to calculate the curvatures of the meridian sections.

OBLIQUE REFRACTION IN A MERIDIAN PLANE

Suppose a wave-surface whose center is at o' and curvature $\underline{c}'$ is incident upon γ at point g; an inspection of the figure will make clear that to reach the point β, s being indefinitely small, it must advance by the distance

$$\tfrac{1}{2}\, s^2 \gamma \cos i + s \sin i - \tfrac{1}{2}(s \cos i)^2\, \underline{c}'.$$

A similar consideration of the refracted wave-surface starting from g will show that, in order to reach β, it must traverse a distance in the direction of its center equal to

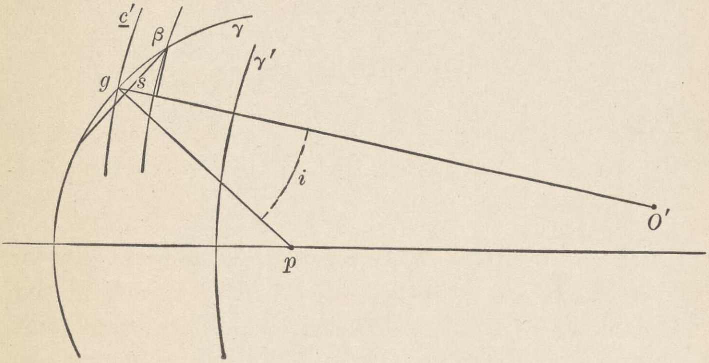

Fig. 13

$$\tfrac{1}{2}\, s^2\gamma \cos r + s \sin r - \tfrac{1}{2}(s \cos r)^2\, \underline{c}_1',$$

if r is the angle of refraction and $\underline{c}_1$ is the curvature of the refracted wave-surface.

The law of refraction makes ρ times the first distance equal to the second; also $\rho \sin i$ equal to $\sin r$. From these equations, therefore,

$$\underline{c}_1 \cos^2 r = \gamma(\cos r - \rho \cos i) + \rho \underline{c}' \cos^2 i.$$

This wave-surface will progress with constantly increasing curvature until it falls after having traversed the distance t_1 on the second refracting surface γ'. Here its curvature, $\underline{c}''$, will be determined by the equation

$$\underline{c}'' = \underline{c}_1(1 - \underline{c}_1 t_1)^{-1} = \mu_1 \underline{c}_1.$$

After refraction at this surface the curvature can be written at once from analogy:

$$\underline{c}_2 \cos^2 r_1 = \gamma'(\cos r_1 - \rho_1 \cos i_1) + \rho \underline{c}'' \cos^2 i_1.$$

This process may be extended to any number of refractions and the general system of equations may be written:

$$\begin{aligned}
\underline{c}_1 \cos^2 r &= \gamma \;(\cos r - \rho \cos i) + \rho \underline{c}' \cos^2 i \\
\underline{c}_2 \cos^2 r_1 &= \gamma' \;(\cos r_1 - \rho_1 \cos i_1) + \rho_1 \mu_1 \underline{c}_1 \cos^2 i_1 \qquad (1) \\
\underline{c}_3 \cos^2 r_2 &= \gamma'' \;(\cos r_2 - \rho_2 \cos i_2) + \rho_2 \mu_2 \underline{c}_2 \cos^2 i_2 \\
&\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
\underline{c}_{\lambda+1} \cos^2 r_\lambda &= \gamma_\lambda (\cos r_\lambda - \rho_\lambda \cos i_\lambda) + \rho_\lambda \mu_\lambda \underline{c}_\lambda \cos^2 i_\lambda
\end{aligned}$$

In cases where the angles of incidence are throughout so small that fourth and higher powers of the angles can be disregarded the equations can be made to take the form:

$$\begin{aligned}
\underline{c}_1 &= \gamma \;(1 - \rho)\{1 + \tfrac{1}{2} \rho i^2 (1 + 2\rho)\} + \rho \underline{c}' \\
&\qquad\qquad \{1 - i^2(1 - \rho^2)\} \\
\underline{c}_2 &= \gamma' \;(1 - \rho_1)\{1 + \tfrac{1}{2} \rho_1 i_1^2 (1 + 2\rho_1)\} + \rho_1 \mu_1 \underline{c}_1 \\
&\qquad\qquad \{1 - i_1^2(1 - \rho_1^2)\} \qquad (1') \\
\underline{c}_3 &= \gamma'' (1 - \rho_2)\{1 + \tfrac{1}{2} \rho_2 i_2^2 (1 + 2\rho_2)\} + \rho_2 \mu_2 \underline{c}_2 \\
&\qquad\qquad \{1 - i_2^2(1 - \rho_2^2)\}
\end{aligned}$$

etc. etc. etc.

OBLIQUE REFRACTION IN A PLANE AT RIGHT ANGLES TO THE MERIDIAN PLANE

In Fig. 14 let β' be the rectangular projection of a point, β, in the refracting surface γ at a very small distance s from the plane of the diagram which is a meridian plane of the refracting surfaces. Suppose a wave-surface incident at g have a curvature $\underline{\bar{c}}'$, its center being at o'. The curvature of a section of this wave-surface by a plane perpendicular to the paper and passing through g and p is $\underline{\bar{c}}' \sec i$. In order that the incident wave-surface may reach the point of which β' is the projection this section must progress along this plane by the distance $\frac{1}{2} s^2\gamma - \frac{1}{2} s^2\underline{\bar{c}}' \sec i$.

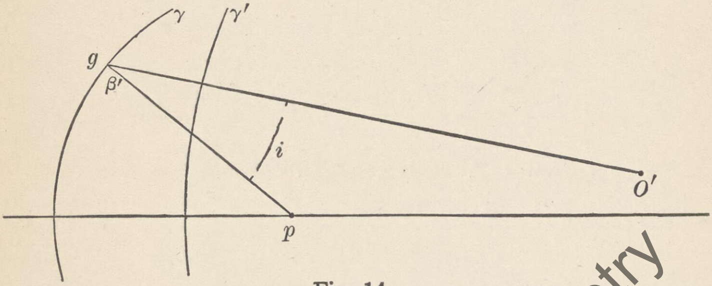

Fig. 14

This displacement of the section is produced by a displacement of the wave-surface towards its center of $\cos i$ times as much, or $\frac{1}{2} s^2\gamma \cos i - \frac{1}{2} s^2\underline{\bar{c}}'$.

A corresponding diagram, in which i and o' are replaced by r and the center of the refracted wave-surface, would yield for the displacement necessary to attain the point β', $\frac{1}{2} s^2\gamma \cos r - \frac{1}{2} s^2\underline{\bar{c}}_1$, if the symbol $\underline{\bar{c}}_1$ represents its curvature.

By the law of refraction, ρ times the first distance equals the second, whence

$$\underline{\bar{c}}_1 = \gamma(\cos r - \rho \cos i) + \rho\underline{\bar{c}}'.$$

This refracted wave-surface will progress with constantly increasing curvature, until, after having traversed the distance t_1, it falls upon a second refracting surface with a curvature given by the equation

$$\underline{\bar{c}}'' = \underline{\bar{c}}_1(1 - \underline{\bar{c}}_1 t_1)^{-1} = \mu_1 \underline{\bar{c}}_1,$$

whence

$$\underline{\bar{c}}_2 = \gamma'(\cos r_1 - \cos i_1) + \rho_1 \mu_1 \underline{\bar{c}}_1.$$

This process may be repeated for any number of successive refractions to which the system of equations below will apply:

$$\begin{aligned} \underline{\bar{c}}_1 &= \gamma\ (\cos r - \rho \cos i) + \rho \underline{\bar{c}}' \\ \underline{\bar{c}}_2 &= \gamma'\ (\cos r_1 - \rho_1 \cos i_1) + \rho_1 \mu_1 \underline{\bar{c}}_1 \\ \underline{\bar{c}}_3 &= \gamma''(\cos r_2 - \rho_2 \cos i_2) + \rho_2 \mu_2 \underline{\bar{c}}_2 \\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \underline{\bar{c}}_{\lambda+1} &= \gamma^\lambda\ (\cos r_\lambda - \rho_\lambda \cos i_\lambda) + \rho_\lambda \mu_\lambda \underline{\bar{c}}_\lambda \end{aligned} \tag{m}$$

In cases where the angles of incidence are throughout so small that fourth and higher powers of the angles may be disregarded the equations may be made to take the following form:

$$\begin{aligned} \underline{\bar{c}}_1 &= \gamma\ (1 - \rho)\{1 + \tfrac{1}{2} \rho i^2\} + \rho \underline{\bar{c}}' \\ \underline{\bar{c}}_2 &= \gamma'\ (1 - \rho_1)\{1 + \tfrac{1}{2} \rho_1 i_1^2\} + \rho_1 \mu_1 \underline{\bar{c}}_1 \\ \underline{\bar{c}}_3 &= \gamma''(1 - \rho_2)\{1 + \tfrac{1}{2} \rho_2 i_2^2\} + \rho_2 \mu_2 \underline{\bar{c}}_2 \\ &\text{etc.} \qquad \text{etc.} \qquad \text{etc.} \end{aligned} \tag{m'}$$

SPHERICAL ABERRATION

Spherical Aberration. — In Fig. 15, let γ be a refracting surface upon which an incident wave-surface of curvature c' falls normally, that is, one whose geometrical center, o, lies somewhere on the axis; also, let gg' represent a meridional section of the refracted wave-surface. This latter surface will not in general be spherical although it will obviously be a surface of revolution. Let o_1 be the image of o' calculated according to the fundamental equations (a); moreover, let o_1' be the point on the axis through which all the light from a narrow zone of radius k will pass, then the distance from o_1' to o_1 is called in the current theory of optical instruments the spherical aberration of the wave-surface having the semi-angular aperture

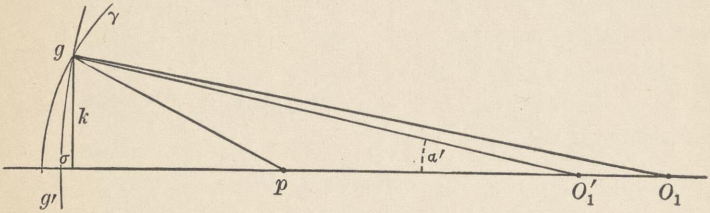

Fig. 15

$g'pg$; its value is, as appears at once from the diagram, equal to $\frac{1}{c_1} - \sigma - \frac{1}{\overline{\underline{c}}_1}\cos\alpha'$. If we restrict our considerations to terms involving nothing higher than the second powers of the small products kc and the like, the relations

$$\sigma = \tfrac{1}{2}\,k^2\gamma,\quad \cos\alpha' = 1 - \tfrac{1}{2}\,k^2\overline{\underline{c}}_1^2$$

are obvious, and the above expression for the spherical aberration becomes by substitution

$$\frac{1}{c_1} - \frac{1}{\overline{\underline{c}}_1} + \tfrac{1}{2}\,k^2(\overline{\underline{c}}_1 - \gamma).$$

The relation of $\overline{\underline{c}}_1$ to c_1 may be deduced from the equations (a) and (m) in the form

$$\overline{\underline{c}}_1 - c_1 = \tfrac{1}{2}\,\gamma\rho(1-\rho)\sin^2 i + \rho(\overline{\underline{c}}_1' - c_1).$$

This can be much simplified by means of the approximations

$$\sin^2 i = k^2(\gamma - c')^2, \ \overline{c}' - c' = \tfrac{1}{2}\, k^2c'^2(\gamma - c'),$$

for then

$$\overline{c}_1 - c_1 = \tfrac{1}{2}\, k^2\{\gamma\rho(1 - \rho)(\gamma - c')^2 + \rho c'^2(\gamma - c')\}$$

Since $\overline{c}_1$ differs from c_1 by terms involving squares and higher powers of k it follows that we may neglect the difference between $\overline{c}_1c_1$ and c_1^2 in the above expressions, as well as that between $\overline{c}_1 - \gamma$ and $c_1 - \gamma$ when either is multiplied by a quantity of the order of k^2c^2. Making such changes the expression for the spherical aberration becomes

$$\tfrac{1}{2}\, k^2\{\gamma\rho(1 - \rho)(\gamma - c')^2 + c_1^2(c_1 - \gamma) + \rho c'^2(\gamma - c')\}\frac{1}{c_1^2}$$

If we indicate by δc_1 the amount that one has to *increase* the curvature of the axial portion of the refracted wave-surface at the vertex in order to have its center o_1 corrrespond with o_1', we have

$$\delta c_1 = \tfrac{1}{2}\, k^2\{\gamma\rho(1 - \rho)(\gamma - c')^2 + c_1^2(c_1 - \gamma) + \rho c'^2(\gamma - c^1)\}$$

From equations (a),

$$\gamma - c' = \frac{1}{\rho}(\gamma - c_1),$$

whence

$$\delta c_1 = \tfrac{1}{2}\, k^2(c_1 - \gamma)^2\left\{\frac{\gamma(1 - \rho)}{\rho} + \frac{c_1^2 - c'^2}{c_1 - \gamma}\right\}$$

By means of equations (a) again, one can derive the relations

$$\frac{\gamma(1 - \rho)}{\rho} = \frac{c_1}{\rho} - c', \text{ and } \frac{c_1^2 - c'^2}{c_1 - \gamma} = -\frac{(1 - \rho)(c_1 - c')}{\rho}$$

whence the above equation takes the form

$$\begin{aligned}\delta c_1 &= \tfrac{1}{2}\,k^2(c_1-\gamma)^2\left(c_1-\frac{c'}{\rho}\right)\\ &= \tfrac{1}{2}\,k^2a_1^2b_1,\end{aligned}$$

where a_1 and b_1 are introduced for convenience in writing.

The quantity δc_1 thus appears as a simple function of the square of the semi-aperture of the refracting surface, hence it is a direct measure of the departure of the refracted wave-surface from the spherical surface having the curvature c_1; for this reason it would be much more logical to give to it the name of spherical aberration than to a quantity erroneously supposed to be due to the sphericity of the refracting surface, whereas spherical surfaces do not differ in this particular from any other surfaces of revolution. In order to escape the introduction of a new word into our text, I shall, therefore, employ the old one in this new sense, remarking that one has only to divide the new expression for the spherical aberration by c_1^2 to find the old conventional value.

At the second surface, where c_1 becomes c'' according to the terminology of equations (a), the difference of curvature becomes:

$$(\delta c_1)'' = \tfrac{1}{2}\,k^2\mu_1^2a_1^2b_1,$$

and after refraction at this surface

$$(\delta c_1)_2 = \tfrac{1}{2}\,k^2\rho_1\mu_1^2a_1^2b_1,$$

so that, after $\lambda+1$ refractions,

$$(\delta c_1)_{\lambda-1} = \tfrac{1}{2}\,k^2\rho_1\rho_2\rho_3\,\ldots\ldots\,\mu_1^2\mu_2^2\mu_3^2\,\ldots\ldots\,a_1^2b_1.$$

In a like manner it may be shown that the aberration produced by refraction at the second surface is

$$\delta c_2 = \tfrac{1}{2}\, k'^2 a_2^2 b_2,$$

which, after refraction at the last surface, becomes

$$(\delta c_2)_{\lambda-1} = \tfrac{1}{2}\, k'^2 \rho_2 \rho_3 \ldots . \mu_2^2 \mu_3^2 \ldots . a_2^2 b_2.$$

The total aberration for the system may be taken as equal to the sum of the aberrations introduced at each surface, that is to

$$\begin{aligned}\Delta c_{\lambda+1} = \tfrac{1}{2}\{ & k^2 \rho_1 \rho_2 \ldots . \rho_\lambda \mu_1^2 \mu_2^2 \ldots . \mu_\lambda^2 a_1^2 b_1 \\ & + k'^2 \rho_2 \rho_3 \ldots . \rho_\lambda \mu_2^2 \mu_3^2 \ldots . \mu_\lambda^2 a_2^2 b_2 \\ & + k''^2 \rho_3 \ldots . \rho_\lambda \mu_3^2 \ldots . \mu_\lambda^2 a_3^2 b_3 \\ & + k^{\lambda 2} \ldots\ldots . a_{\lambda+1}^2 b_{\lambda+1}\}\end{aligned}$$

This may be easily transformed to the more convenient form

$$\begin{aligned}\Delta c_{\lambda+1} = \tfrac{1}{2}\, k^2 \mu_\lambda^2 !\, \rho_0 \{ & \rho^{-1} a_1^2 b_1 + \rho_1^{-1} !\, \mu_1^{-4} !\, a_2^2 b_2 + \\ & \rho_2^{-1} !\, \mu_2^{-4} !\, a_3^2 b_3 + \ldots . + \rho_\lambda^{-1} !\, \mu_\lambda^{-4} !\, a_\lambda^2 b_\lambda \}\end{aligned} \qquad \text{(n)}$$

In this equation the aberration is that of the emergent wave-surface expressed in terms of physical constants of the system and of the curvature of the incident wave-surface at the first refracting surface. It is often convenient to have the values for other points of reference; such can be easily found by the general transformation equations (d). For example, suppose the first and second principal points be chosen for reference, K being the semi-diameter of the zone at the first principal plane corresponding to k at the first refracting surface; then, since,

$$\begin{aligned} C_1 &= c_{\lambda+1}(1 - \xi' c_{\lambda+1})^{-1} \\ K &= k(1 - \xi c'),\end{aligned}$$

we have

$$\Delta C_1 = \tfrac{1}{2} K^2(1 - \xi c')^{-2}(1 - \xi' c_{\lambda+1})^{-2}\mu_\lambda^2!\, \rho_0\{\rho^{-1}a_1^2 b_1 + c_1^{-1}!\, \mu_1^{-4}\,!\, a_2^2 b_2 + \dots\dots + \rho_\lambda^{-1}!\, \mu_\lambda^{-4}!\, a_\lambda^2 b_\lambda\} \quad (n')$$

Zonal Differences of Magnification

Consider the ratio of the size of the image o_1 to that of the object o' when, as illustrated in Fig. 16, the image is formed by the light transmitted through a small area of the refracting surface at a distance k from the axis. This magnification is, according to equation (c) of page 9, expressed by

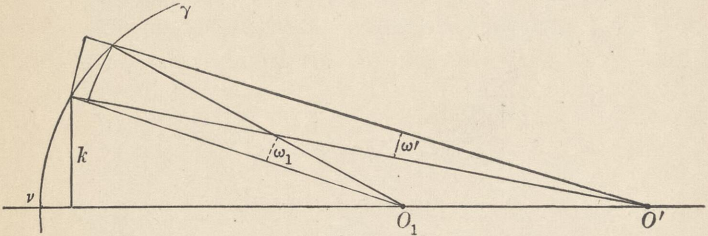

Fig. 16

$$m = \frac{o_1}{o'} = \rho\,\frac{\sin\omega'}{\sin\omega_1} = \rho\,\frac{\cos i}{\cos r}\,\frac{\underline{c}'}{\underline{c}_1}.$$

At the second refraction the ratio of image to object is, similarly,

$$m' = \rho_1\,\frac{\cos i_1}{\cos r_1}\,\frac{\underline{c}''}{\underline{c}_2};$$

hence, in the case of $\lambda + 1$ successive refractions we may write

$$M = mm'm'' \dots\dots m^\lambda = \rho_0\,\frac{\cos i\,\cos i_\lambda \dots\dots \cos i_\lambda}{\cos r\,\cos r_1 \dots\dots \cos r_\lambda}\;\frac{\underline{c}'\underline{c}'' \dots\dots \underline{c}^{\lambda+1}}{\underline{c}_1\underline{c}_2 \dots\dots \underline{c}_{\lambda+1}}.$$

If the zone is indefinitely near the axis, in other words, if all values of k are very small, the magnification becomes

$$M_0 = \rho_0 \frac{c'c'' \ldots\ldots c^{\lambda+1}}{c_1 c_2 \ldots\ldots c_{\lambda+1}} = \rho_0 \mu_\lambda! \frac{c'}{c_{\lambda+1}},$$

which agrees, as it should, with the value found for the same ratio on page 5.

The above value of M can be greatly simplified when the angles of incidence are so small that powers of the angle higher than the second are insignificant, for then

$$\begin{aligned} \cos i &= 1 - \tfrac{1}{2} i^2 \\ \cos r &= 1 - \tfrac{1}{2} \rho^2 i^2 \end{aligned} \qquad \frac{\cos i}{\cos r} = 1 - \tfrac{1}{2} i^2(1 - \rho^2)$$

and so on for the following surfaces. Further, if we consider only cases in which all terms involving kc and the like with powers higher than the square may be safely ignored, we have also

$$\begin{aligned} \underline{c}'' &= \mu_1 \underline{c}_1 \\ \underline{c}''' &= \mu_2 \underline{c}_2 \\ &\text{etc., etc.} \end{aligned}$$

whence,

$$M = \rho_0 \mu_\lambda! \left\{1 - \tfrac{1}{2} i^2(1 - \rho^2) - \tfrac{1}{2} i_1^2(1 - \rho_1^2) - \ldots\ldots - \tfrac{1}{2} i_\lambda^2(1 - \rho_\lambda^2)\right\} \frac{\underline{c}'}{\underline{c}_{\lambda+1}}.$$

The ratio $\frac{M}{M_0}$ is then defined by the equation

$$\frac{M}{M_0} = \left\{1 - \tfrac{1}{2} i^2(1 - \rho^2) - \tfrac{1}{2} i_1^2(1 - \rho_1^2) - \ldots\ldots - \tfrac{1}{2} i_\lambda^2(1 - \rho_\lambda^2)\right\} \frac{\underline{c}'}{c'} \frac{c_{\lambda+1}}{\underline{c}_{\lambda+1}}$$

Within the limits of precision fixed above we have, in regions near the axis,

$$\frac{\underline{c}'}{c'} = 1 + \tfrac{1}{2}k^2(\gamma - c')c'$$

$$\frac{c_{\lambda+1}}{\underline{c}_{\lambda+1}} = 1 - \tfrac{1}{2}k_\lambda^2(\gamma^\lambda - c_{\lambda+1})c_{\lambda+1}$$

$$i = k(\gamma - c');\ i_1 = k'(\gamma' - c''), \text{ etc.}$$

Making the substitutions and carrying out the multiplication with omission of terms containing powers of k higher than the second

$$\frac{M}{M_0} = 1 - \tfrac{1}{2}k^2(\gamma - c')^2(1 - \rho^2) - \tfrac{1}{2}k'^2(\gamma' - c'')^2(1 - \rho_1^2) - \text{etc.}$$
$$+ \tfrac{1}{2}k^2(\gamma - c')c' - \tfrac{1}{2}k_\lambda^2(\gamma^\lambda - c_{\lambda+1})c_{\lambda+1}.$$

The number $\frac{M - M_0}{M_0}$ may be called the *zonal coefficient of magnification:* this number will be designated hereafter by Z. When k', k'', etc., are expressed in terms of k and $\gamma - c'$, $\gamma' - c''$, etc., in terms of a_1, a_2, etc., the final expression for the zonal coefficient of magnification becomes

$$Z = \tfrac{1}{2}k^2\{[\mu_\lambda^{-2}!\, a_\lambda c_{\lambda+1} - a_1\rho^{-1}c']$$
$$+ [a_1^2(1 - \rho^{-2}) + \mu_1^{-2}!\, a_2^2(1 - \rho_1^{-2}) + \dots$$
$$+ \mu_\lambda^{-2}!\, a_\lambda^2(1 - \rho_\lambda^{-2})]\} \quad \text{(o)}$$

To express this in terms of K it is only necessary to employ the equation

$$k = (1 - \xi c')^{-1}K,$$

when ξ is the distance from the first vertex to E.

CHAPTER V

IMAGE REMOTE FROM THE AXIS — CURVATURE OF IMAGE SURFACE — ASTIGMATISM — DISTORTION

Curvature of Image Surface. — Suppose Figure 7 represents an optical system with its first vertex at v and its last at v^λ, the first and second principal points being at E and at E_1, respectively; then, if o' is the source of light the place of its image, o_1, can be found by means of the general equations (a). Now imagine o' displaced perpendicularly to the axis by a small distance, the image o_1 will also be displaced along a path departing from the axis at right angles and with the same angular velocity, but the path will in general be curved. The curvature of this line we shall take as the measure of the *curvature of the image surface*. If one knows the angles of incidence of that portion of the incident wave-surface which passes through E it is easy to find the curvature of this portion by use of the equations (l) of the last chapter. If these requi-

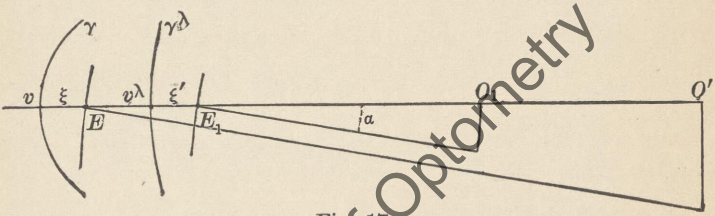

Fig. 17

site angles are small throughout, one may find them very readily by means of the general equations (a), as appears from the following:

Assume E as a source of light; then, with c' equal to $\frac{1}{E}$, follow through the calculations of (a); the final value, $c_{\lambda+1}$, will clearly be equal to $\frac{1}{E_1}$, since E_1 is the image of E. Now set $k = \xi \tan \alpha$, then the distances from the axis to the successive points of incidence of this definite portion of the incident wave-surface will be

$$\begin{array}{lll} k = \xi tg\alpha & & i = k(\gamma - c') \\ k' = \mu_1^{-1}k & \text{Also} & i' = k'(\gamma' - c'') \\ k'' = \mu_2^{-1}!\,k & & i'' = k''(\gamma'' - c''') \\ \text{etc., etc.} & & \text{etc., etc.} \end{array}$$

With these determined values of i we may calculate the successive values of $\underline{c}_1$, $\underline{c}_2$, etc., by means of the equations (e′) and compare the final value with that of the axial wave-surface: or, still better, we may proceed as follows:

Suppose that ϕ, with the appropriate suffix, represents the increment of curvature at any refraction due to the obliquity alone, or in other words, suppose at that particular refraction we ignore the difference in curvature of the oblique and axial wave-surfaces; then by subtracting the equation in group (a) from the corresponding equation in (l′), we may write

$$\begin{array}{l} \phi_1 = \{\gamma(1 - \rho)(\tfrac{1}{2} + \rho) - c'(1 - \rho^2)\}\,\rho i^2 \\ \phi_2 = \{\gamma'(1 - \rho_1)(\tfrac{1}{2} + \rho_1) - c''(1 - \rho_1^2)\}\,\rho_1 i'^2 \qquad \text{(p)} \\ \text{etc.} \quad \text{etc.} \quad \text{etc.} \end{array}$$

These values may be summed with attention to the conditions, first, that any such difference grows continuously during propagation so that at the next refracting surface it will have increased in the ratio μ^2; second, that at this surface the value is changed suddenly in the ratio of the velocity of propagation in the succeeding medium to that in the preceding. Thus we may write finally

$$\underline{c}_{\lambda+1} - c_{\lambda+1} = \phi_{\lambda+1} + \mu_\lambda^2 \rho_\lambda \phi_\lambda + \mu_{\lambda-1}^2 \mu_\lambda^2 \rho_{\lambda-1} \rho_\lambda \phi_{\lambda-1} + \ldots . \\ + \mu_\lambda^2 ! \, \rho_1 \ldots . \rho_\lambda \phi_1 \qquad (\mathrm{p}')$$

This quantity may be referred to the points E and E_1, respectively, by employing the equations of transformation so that

$$\underline{C}_1 - C_1 = (1 - \xi' c_{\lambda+1})^{-2} (\underline{c}_{\lambda+1} - c_{\lambda+1}) \qquad (\mathrm{p}'')$$

The departure of the image surface from the perpendicular plane passing through F is

$$-(\underline{C}_1 - C_1) C_1^{-2}.$$

From the equations above giving the values of i, i', etc., it is obvious that this last value increases as the square of the tangent of the angular displacement from the axis α.

Astigmatism.— In general the curvature of the refracted wave-surface is not strictly spherical, but at any point one of its two principal values, necessarily at right angles to each other, lies in the meridian plane. The value of this one has just been given in the preceding section, that of the other is found on page 74 in the group of equations (m′). If now, following exactly the course described in the preceding article, each value in (m′) is subtracted from the corresponding value in (l′) and the difference, ignoring as in that case the differences between the two incident curvatures, is indicated by ψ with the appropriate suffix, we may write

$$\begin{aligned} \psi_1 &= \{\gamma(1-\rho)\rho - c'(1-\rho^2)\}\rho i^2 \\ \psi_2 &= \{\gamma'(1-\rho_1)\rho_1 - c''(1-\rho_1^2)\}\rho_1 i'^2. \\ \text{etc.} & \quad \text{etc.} \quad \text{etc.} \end{aligned}$$

From these we may at once deduce, in accordance with the reasoning developed above:

$$\underline{c}_{\lambda+1} - \bar{\underline{c}}_{\lambda+1} = \psi_{\lambda+1} + \mu_\lambda^2\rho_\lambda\psi_\lambda + \mu_{\lambda-1}^2\mu_\lambda^2\rho_{\lambda-1}\rho_\lambda\psi_{\lambda+1} + \ldots \\ + \mu_\lambda^2!\,\rho_1\ldots\ldots\rho_\lambda\,\psi_1.$$

The quantity $\dfrac{\underline{c}_{\lambda+1} - \bar{\underline{c}}_{\lambda+1}}{-c_{\lambda+1}^2}$ is defined as the astigmatism due to obliquity: its value increases as the square of the angle α.

Distortion. — Imagine any optical system having a centered diaphragm, as at d in Fig. 18; this will have

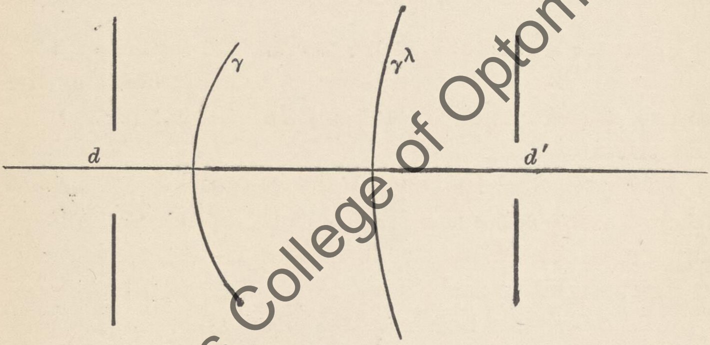

Fig. 18

its image, say at d', which is determinable in position and magnitude from the constants of the system.

This is a characteristic of almost all optical instruments; for example, in the telescope the cell of the objective constitutes the first diaphragm, its image being the ocular circle; in a lens employed as a simple magnifier the corneal image of the pupil of the observer's eye is the first diaphragm; in the compound microscope the first diaphragm is ordinarily the virtual image of the last lens of the objective; in the photographic camera it is generally the virtual image by the anterior portion of the system of the iris diaphragm or its substitute.

For the moment let us suppose that the system is such that the image of a point at the center of d is formed at the center of d' by refractions without ultimate spherical aberration; or in other words, that all spherical wave-surfaces in the first medium which have their common geometrical center at the middle point of d become spherical wave-surfaces in the final medium with a common center at the middle point of d'; then it follows that every imaginary point or line on a spherical wave-surface before refraction has a corresponding point or line on all refracted wave-surfaces, which lines are geometrically similar to their imaginary origins. It is easy to extend this conception to the conclusion that the images of all objects in the first medium will be geometrically similar to the objects, provided that only such portions of the incident wave-surfaces are involved as pass close to the center of d, or what amounts to the same thing, if the aperture in the diaphragm is sufficiently small. Indeed, if the aperture is very minute in comparison with the distance of the nearest object the optical system may be suppressed altogether; in this case *d′* will correspond with *d* and the apparatus, properly shielded, becomes the "pin-hole camera" in which the perfection of delineation is only limited by the magnitude of the light waves.

As the image of a straight line formed by a lens system possessing the properties just described always remains a straight line, such a system is styled *rectilinear*. It is obvious that the condition of rectilinearity may be very important in certain cases, notably in camera systems employed for architectural photography or for copying diagrams, while in those cases where only images close to the axis are useful this feature may often be advantageously sacrificed for others of more immediate importance for the ends in view, such as brightness or even simplicity of construction.

The manner in which the error under discussion presents itself to the user of a photographic camera is

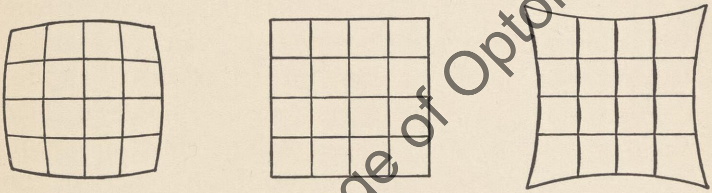

Fig. 19

illustrated in Fig. 19, where the middle diagram represents the image of a rectangular grating when the particular type of spherical aberration described above vanishes, that at the left represents the image of the same object when the aberration is positive, and the remaining diagram when the aberration is negative.

The appropriateness of the current term *distorsion* to designate the error is obvious; not so, the rather fantastic terms "barrel-form" and "cushion-form" distorsion to distinguish the two types of error which have been used by some writers. It would appear better to call the former type — by far the more familiar one since it is always present when a simple lens is used as a magnifier — positive distorsion and the latter negative distorsion.

CHAPTER VI

TELESCOPES — MICROSCOPES — OCULARS — CAMERA SYSTEMS

As an illustration of the application of the theorems developed in the preceding chapters we shall consider in the present one a number of types of telescopes, treating the objectives and oculars as separate systems. A sufficient number of each will be treated in detail so that any other form would easily find an appropriate method by extension.

As a first example we may discuss the famous Washington twenty-six-inch telescope made by the Clarks in 1873 and with which Professor Asaph Hall discovered the satellites of Mars. This particular instrument is chosen not only because it is one of the best of the telescopes made by its famous constructors but because pure empiricism was regarded by them as a wholly sufficient guide, and it will be highly instructive to find how near such a method led to the best results. Unfortunately we know no better than the makers did the optical constants of the materials employed, but we do know far better than they did, thanks to the investigations of Professor Holden, the radii, thicknesses and focal length; moreover, I have in my possession a "color curve" for the telescope as derived from the measurements made by Professor Brown in 1899 at my request when asked to design a corrector for certain spectroscopic uses. This appears below on page 93.

The problem in its simplest aspect is to find materials which satisfy the following conditions — neglecting small modifications due to thicknesses of the lenses:

(a) $C_0 = A(n_0 - 1) + B(n'_0 - 1)$ Prescribed focal length

(b) $\left(\dfrac{dc}{dn}\right)_0 = 0 = A + B\alpha$ Condition of achromatism

(c) $\left(\dfrac{d^2c}{dn^2}\right)_0 = 2B\beta$ Color curve

Here A and B are the curvature sums for the crown and flint lenses, respectively, n_0 and n_0' the indices of refraction of the materials corresponding to the minimum focal length. The indices are connected by the equation

$$n' - n'_0 = \alpha(n - n_0) + \beta(n - n_0)^2.$$

Professor Brown's measurements show that n_0 and n'_0 are exactly midway between n_C and n_F, which implies a very small undercorrection for color, according to the standard advocated in Chapter III, but not greater than is found in other of Clark's objectives.[1]

It appears as though we have four unknowns to be determined from only three equations and that the solution is therefore indeterminate; but this difficulty disappears when one recognizes that with determinate types of glasses the ratio of β to α is also narrowly fixed, while we know that only ordinary silicate glasses could have been used at the time the objective was made.

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[1] See Amer. Jour. Sci., Vol. 23, 1882, p. 167.

An application of the method led to the disappointing result that no material experimentally known to me, nor others which were described in maker's catalogues, would yield the known values. It was difficult to escape the conclusion that the accepted radii were wrong — especially as Holden had himself raised the question of a possible flexure in the flint lens from unsuitable support. This led me to write to the Superintendent of the Naval Observatory, Captain Hoogewerff, U.S.N., asking if he would have the system of Newton's rings from the inner surface reflections observed by means of a sodium flame. This he kindly did, with the result that the third surface was demonstrated as not having a radius much more than 0.1 inch longer than that of the second — a difference which is practically negligible and which may possibly also be due to flexure. We may be sure, therefore, that Clark formed the first, second, and third surfaces upon the single pair of tools and we may amend Holden's radii as follows:

	Holden		*Adopted*	
r_1	$+ 161.39$		$+ 161.39$	
r_2	$- 161.39$	$t_1 = 1.884$	$- 161.39$	$t_1 = 1.884$
r_3	$- 162.07$	$t_2 = 0.016$	$- 161.39$	$t_2 = 0.000$
r_4	$- 19466$	$t_3 = 0.958$	$- 39000$	$t_3 = 0.958$

The fourth radius is deduced from the assumption that the flexure was equal in this measurement to that obtaining in the measurement of the third surface.

Of the several hundreds of specimens of optical glass which I have measured with precision two, and two only, yield a solution which may be regarded as of satisfactory closeness. These are flint 1237 of Feil et Cie. and crown 4341 from the same source.*

The measured indices of the crown are

Line	n
C	1.515483
λ 5.614	1.519327
F	1.524071
G	1.529211
h	1.531821

$$n_0 = 1.51978$$

$$n_0' = 1.62889$$

$$n' - n_0' = 2.011(n - n_0) + 11.74(n - n_0)^2$$

From these values, since we may imagine the two lenses in contact, the following table may be derived:

	$\log \rho$	$\log (1 - \rho)$	$\dfrac{\log d\rho}{dn}$.		$\log a$	$\log b$
( )	9.81822	9.53404	9.6364N	(1)	7.6114N	7.3272
(1)	9.96989	8.82575	9.7309N	(2)	7.8913	6.8524N
(2)	21189	9.79857N	3035	(3)	7.4166	7.2085

These constants applied in equations (a) and (k) yield:

c_1	μ_1	c_2	μ_2	c_3
.002124	1.0040	.001574	1.0015	.005842

$$P = .005700 \qquad P^{-1} = 389.2$$

---

* It is not surprising that in several hundred specimens only two should meet the requirements; on the contrary, the surprise lies in the fact that any such close correspondence should be found. Indeed, I am disposed to think that flint 1237 may be actually the material which entered the Washington object glass. A considerable quantity of it was supplied to me by Feil in 1875 as the best of the type in his possession, hence it was probably made before the objective and had an exceptional reputation. The constants of this glass, transcribed in part in Appendix B, are given in great detail in the American Journal of Science, Vol. XV, 1878, p. 269. The crown glass is that employed in the 12-inch objective of 1800 inches focal length of the Mt. Wilson Observatory.

The measured value of P^{-1} is 389.66 with a probable error of ± 0.26 as calculated by Holden, while the chromatic aberration indicates a longer focal length for light of refrangibility of F by 0.011, which is as nearly in agreement with measurement as could be looked for.

The method developed on page 59 enables us at once to calculate the secondary error in color of which Professor Brown has given the values; they are:

	C	*D*	(*o*)	*b*	*F*	*G*	*h*
Measured	0.23	0.09	0.00	0.09	0.24	0.95	1.58
Calculated	0.20	—	0.00	—	0.20	1.00	1.60

The two color curves are practically identical.

The equations (n) of Chapter IV yield:

$$\Delta c_3 = \tfrac{1}{2} k^2(-\,5.5)\ (10^{-9}),$$

whence the spherical aberration is $-\,0.05$, an amount much less than is ordinarily due to processes of construction and which the optician corrects by local polishing.

Microscopes. — The method for calculating the proper constants for microscope objectives needs no alterations from that employed for telescopes, although the general forms must be greatly varied when high powers are required. For a magnification from twenty to forty a properly designed achromatic doublet is satisfactory, while for magnifications from forty to one hundred, the form invented by Lister is universally employed. For still higher powers a single or double front of crown glass is necessary. The modern apochromatic objective necessitates the use of other materials than ordinary silicate glasses.*

Oculars. — These must be regarded as parts of a telescope or microscope, not as independent systems. When adjusted for vision in either of these instruments the incident wave-surfaces are narrowly limited; in other words, the values of k in the preceding formulas are always small. To serve as an illustration the essential constants of the familiar Huyghenian ocular may be given here, referring for more detailed information concerning the properties of various types to the article cited on page 56.

The conditions assumed for definiteness are, that the ocular is used with a telescope of considerable length with the ordinary ratio of fifteen to one as that of the focal length to aperture; that its focal length is one inch; and, finally, that the material of the lenses is the crown glass described on page 92.

It is obvious that all the light which is transmitted by the ocular passes through the objective and also through the image of the objective which is a circular area one fifteenth of an inch in diameter — the eye-circle.

The constants of such an ocular are the following, the positions of the cardinal points being measured from the first vertex of the anterior or field lens:

r_1	r_2	r_3	r_4	E	E'	F	F'
1.0395	∞	0.347	∞	0.652	2.000	1.652	1.000

---

* An account of the history of the microscope with its various types of construction of the objectives may be found in the Yale Bicentennial volume entitled "Light, a Consideration of the More Familiar Phenomena of Optics," by the present writer.

The color error, that is, the chromatic variation of F', is 1.8 times as great as that of a single lens of the same material while the spherical aberration is 9.6 times as great as that of a double convex lens. Both of these errors are far from inconsiderable and make this type of ocular distinctly inferior.

Photographic Camera Lenses. — Just as the wave-surfaces in a telescope have two circular boundaries, namely, the objective and the eye-circle, the camera system has two which are the virtual images of the diaphragm produced by the anterior and posterior lenses, respectively. Beyond this property which is common to all, there is a very great variety of types to meet the varying requirements. For convenience three classes may be specified. First, where sharp definition over a moderate angular field accompanied by greatest attainable brightness of image is demanded, nothing is better than the famous Portrait Lens invented by Petzval and which still bears his name. The image surface is somewhat strongly curved and for points remote from the axis there is pronounced astigmatism, features which restrict the extent of the useful field. Second, when rigid geometrical reproduction is demanded while maximum brightness is not essential, as in copying cameras, the symmetric construction invented by Steinheil, and very extensively imitated, occupied this field for many years. It still maintains an important position in practice. Third, with the recent introduction of barium crown glasses, which are characterized by high refractive power with accompanying low dispersive power, and an enormously increased use of photography has come an almost endless variety of new forms. To do more than mention the fact would take us too far from the immediate purpose of this writing, but it should be noted that Petzval had pointed out in his important theoretical discussion published in 1840 that, if a glass of higher refractive power than the ordinary crown glass without increased dispersive power were available, the image surface could be greatly flattened without excessive oblique astigmatism.

Two considerations of practical importance may be added. When the best definition for point sources is demanded, as in astronomical use, the color correction should be adapted to the nature of the sensitive plate employed; but for general use where a separation of the visual and photographic images is inconvenient the chromatic correction should be made for an intermediate color — that of the Fraunhofer line *F*, for example. The second consideration relates to the method of applying the formulas of the preceding pages to a camera system. A method found satisfactory is to calculate the errors of the anterior system first as though it existed alone, then, taking into account the limitations imposed by the restricting diaphragm, find how these errors are modified by the action of the remaining lenses.

APPENDIX A

The important relation expressed by (c) has been proved for very small values of the angles only, and it would therefore hold good if either ω or $tg\omega$ were substituted for $\sin\omega$; indeed, the relation in terms of $tg\omega$ was first established by Lagrange and was a most important contribution to the theory of optics. It is possible to show, however, that the form (c) is correct for all values of ω' provided that both incident and finally refracted wave-surfaces are spherical; in other words, provided that the system is free from spherical aberration. To do this, let pq, Fig. 20, represent the incident wave-surface c' and $p'q'$ the finally refracted wave-surface $c_{\lambda+1}$.

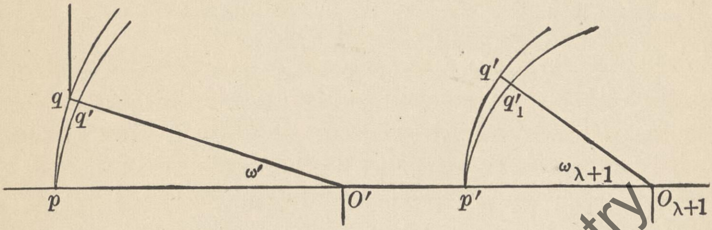

Fig. 20

If c' is limited by a diaphragm at q, then $c_{\lambda+1}$ is also limited at the point q' which is the *corresponding* point to q, that is to say, the point in $c_{\lambda+1}$ where all the light energy comes from a region in c' indefinitely near q. The semi-angular aperture of the incident and finally refracted wave-surfaces are, respectively, ω' and $\omega_{\lambda+1}$ as appears in the figure. Now suppose the incident wave-surface to be inclined by an indefinitely small angle $o'c_1'$, then the finally refracted wave-surface will also be inclined by an indefinitely small angle at p' equal to $o_{\lambda+1}c_{\lambda+1}$. It is apparent from the figure that in the oblique wave-surface q_1 is a corresponding point to q and q_1' to q', since the waves are propagated in the direction of the normals to their surfaces, hence q_1' is a corresponding point to q_1. Moreover, since p and p' are corresponding points in the two wave-surfaces, the time required for light to pass from q to q' and from q_1 to q_1' is, in each case, equal to the time required to pass from p to p'; hence the time required for progression from q to q_1 is equal to that from q' to q_1'. The velocity of propagation in the last medium, however, is ρ_0 times as great as in the first; consequently we have

$$\rho_0 q q_1 = q' q_1'.$$

By inspection of the figure we see that

$$\begin{aligned} q q_1 &= pq \cdot o'c' \cos \tfrac{1}{2}\,\omega' \\ q' q_1' &= p'q' \cdot o_{\lambda+1} c_{\lambda+1} \cos \tfrac{1}{2}\,\omega_{\lambda+1} \end{aligned}$$

Combining these three equations and substituting the trigonometrical expressions for the chords pq and $p'q'$ we find

$$2\,\rho_0 o' \sin \tfrac{1}{2}\,\omega' \cos \tfrac{1}{2}\,\omega' = 2\,o_{\lambda+1} \sin \tfrac{1}{2}\,\omega_{\lambda+1} \cos \tfrac{1}{2}\,\omega_{\lambda+1},$$

whence we derive immediately the equation (c).

This highly important principle is essential in determining the absolute power of every optical system. Its truth was assumed by Professor Abbe in his celebrated paper on the limit of power in the microscope and it was shown as a consequence of the second law of thermodynamics by Helmholtz in a contemporaneous paper on the same subject.

APPENDIX B

INDICES OF REFRACTION UTILIZED IN TEXT

Line	Cr. 4341	Fl. 1237	Cr. 339*	Fl. 0.161†
A	——	1.615258	1.518149	1.561019
B	——	1.618706	1.520103	1.563611
C	1.515483	1.620482	1.521047	1.564947
D_2	——	1.625421	1.523719	1.568532
λ 5614	1.519327	1.628001	1.525073	1.570344
E	——	1.631893	1.527100	1.573070
λ 5006	——	——	1.528932	1.575539
F	1.524071	1.637756	1.530069	1.577103
G	——	1.649086	1.535623	1.584719
h	1.531821	1.654848	1.538350	1.588448

* Type of 0.13 of Jena catalogue.

† In the catalogue n_D is given as 1.5676 which is certainly erroneous and probably a misprint.

APPENDIX C

*Irregular Atmospheric Refractions — Mirages.** — The principle developed on pages 8 and 9, and which is established mathematically in Appendix A, admits an interesting extension. Suppose that a portion of the wave-surface from a distant point be bounded by an imaginary circle as at 1 in Fig. 21. At a distance twice as great the imaginary boundary will have doubled in size and, according to this principle, the eye placed at the second point will see the source and its immediate surroundings as having just half the angular dimensions as they would appear to have seen from the nearer point.. This is, of course, in accord with geometrical perspective. Again, suppose that the intervening medium had so modified the boundary that it takes the form 3; then the object would appear from this point to have its angular dimensions in a vertical direction increased relatively in the ratio of the longer diameter of the boundary to that of the shorter. Such a modification is frequently produced by passage through the atmosphere

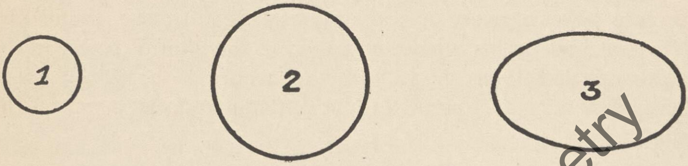

Fig. 21

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* The current explanations of these curious phenomena are far from satisfactory. The older one supposes that part of the light from a distant object near the horizon is totally reflected from a layer of rarer air near the ground, an explanation which must be unconditionally rejected since it implies the existence of a surface of discontinuity, like that separating air and water for example, which is clearly impossible. The other explanation assumes that part of the light is refracted in the lower strata of warmer air so that a second, depressed image of the object becomes visible. This is much nearer the true explanation but fails to show whether the extra image is inverted, which may or may not be the case; only in the former does it represent the desert mirage.

close to the surface of the earth when there is appreciable difference in temperature there and in the air above. Temperature differences of this kind give rise to highly varied phenomena which are classed under the general name of Mirages. Before turning to consideration of the more striking cases it is important to note that only distant objects close to the horizon are thus affected under natural conditions, and, though common enough, they are ordinarily overlooked. When looked for, especially when the observer is aided by a telescope of moderate power or even by an opera-glass, there will be no difficulty in detecting over extended plains or the sea the types of particular interest. The prevailing misconceptions attaching to these interesting phenomena are largely due to the neglect of writers who have described extraordinary cases to make clear to their readers the actual minuteness of the strange apparitions.

The most common cases, in temperate latitudes at least, is where a stratum of air close to the ground, or to the surface of a large body of water, is at a higher temperature than that prevailing above. This condition may be produced either by the high temperature of the sun-heated soil, or, especially over bodies of water, by a nocturnal cooling of the lower atmosphere by radiation. The latter condition is very apt to prevail during quiet morning hours after a cloudless night in late summer. It is to be noted that in these cases the lighter air below is in a state of unstable equilibrium with the denser air above and that this instability tends towards fitfulness, both in duration and in direction, in the visual phenomena. This explains also why over the warmer surface below the mirage is usually limited in azimuth. For example, over a heated plain it is very usual to see the appearance of a lake in some one direction but the observer probably never finds himself surrounded by the appearance of water on all sides at once.

There is another element which should be noted before attempting a general explanation, and that is the influence of the curvature of the earth's surface when distances of thousands of feet are involved. The obvious effect of this curvature is to restrict the region where the wave-surfaces undergo modifications so that, in general, both the object and the observer are above the layer of abnormally heated air.

In Fig. 22 let *ab* represent the surface of the earth or of the sea in the region where the abnormal refraction takes place, *A* the place of an object of limited angular magnitude in a vertical direction and *B* the position of the observer. This, of course, is a special case but it is that of the desert mirage and can readily be extended to other less striking cases.

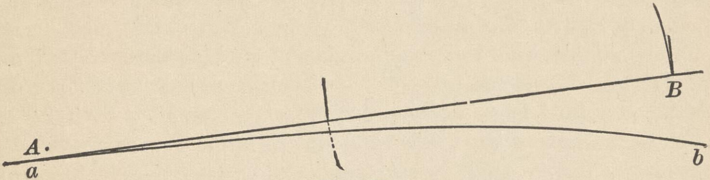

Fig. 22

Noting that the horizontal distances in the diagram represent thousands of feet while the vertical distances would be reckoned in inches, it is easy to see what form of changes the wave-surfaces undergo. After the wave-surfaces enter the region of modification the upper portions move on without abnormal changes while the lower portions move with a greater velocity in the rarer medium below, the general result being that the wave-surfaces above the warmer air are continuously approaching flatness and in it they are becoming cylindrical with progressively increasing curvature. If the observer is so placed that only these cylindrical portions can reach him, the object and its immediate surroundings are slightly depressed and magnified in a vertical direction — a very common occurrence. Further on the cylindrical wave-surfaces will have emerged from the stratum of heated air and be subject afterwards to only normal changes of curvature; but this results finally in a reversal of curvature so that a wave-surface originating at *A* has two sheets, both essentially flat on account of the great distances involved. The interpretation of the visual effect is evident. The object *A* will be seen in ordinary aspect with an inverted image below but slightly distorted. The extraordinary inverted image is obviously bounded below by the line of the sensible horizon. If *A* is on the sky line its inverted image is at the edge of an image of the sky and this simulates a body of water limited in the foreground by the sensible horizon. It is obvious that this sensible horizon moves with the observer, whence follows the fugitive aspect of the desert mirage. If the distant object is a ship an inverted image may be seen below it although the horizon line usually cuts off all except the portions near the water unless the angular height of the vessel is very small.

The description of the more familiar type of mirages renders that of the second kind much easier. These cases are usually produced by an accumulation of warm air on calm summer days over a cold sea, or in arctic regions, where some of the most remarkable observations have been recorded, over fields of ice. Since the heated air is above the cold and denser air there is no such markedly unstable equilibrium as in the cases discussed above and they are generally less evanescent; in addition, there is no obvious reason why the upper limit of the critical stratum might not lie well aloft. Some very striking mirages of this kind off the coast of Maine and over Bay of Fundy waters have been seen by the writer.

Fig. 23 will help to clarify the following explanation.

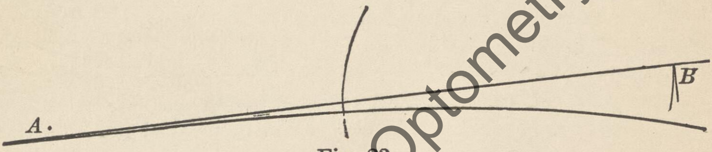

Fig. 23

Represent the source at *A* and the observer at *B* with the stratum of cold homogeneous air between the line and the sea. Above the former line the light will move faster in the lighter air so that the wave-surfaces become cylindrically concave forward and, under certain circumstances, will reverse their curvature before reaching the observer, as indicated in the figure. In this case the distant object will appear in its natural form, or, possibly, slightly altered vertically, surmounted by an inverted image little distorted. The phenomenon is much more apt to excite the attention of a spectator since there is no horizon line to restrict the extent of the inverted image as in the complementary case of warm air below.

It is conceivable that the upper portions of the wave-surfaces, where the curvature is less abrupt on account of lessened temperature gradient, might also reach the observer at *B* without reversal with a consequent third erect image above. Whether such a complication has been recorded is unknown to me, but at any rate it necessarily follows that, to an observer at an altitude well above *B*, a distant object, even one well below the sensible horizon under ordinary conditions, might be visible with a certain amount of extension. Such observations are by no means rare, the mariner's term for the phenomena being "looming." The occasional visibility of the coast of France from the shores of England affords an illustration.